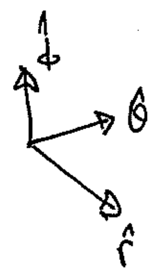
9.17 Låt S vara rotationsellipsoiden $r = \dfrac{1}{2-\cos\theta}$.

S parametriseras av ortsvektorn

$$\bar{r}(\theta,\phi) = \frac{1}{2-\cos\theta}\,\hat{r}$$

$$\bar{r}'_\theta \times \bar{r}'_\phi = \left[\frac{-\sin\theta}{(2-\cos\theta)^2}\,\hat{r} + \frac{1}{2-\cos\theta}\,\hat{\theta}\right] \times \frac{\sin\theta}{2-\cos\theta}\,\hat{\phi}$$

[ty $\dfrac{\partial}{\partial\theta}\hat{r} = \hat{\theta}$, $\dfrac{\partial}{\partial\phi}\hat{r} = \sin\theta\,\hat{\phi}$]



$$= \frac{\sin\theta}{(2-\cos\theta)^2}\,\hat{r} + \frac{\sin^2\theta}{(2-\cos\theta)^3}\,\hat{\theta}$$

med $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. Visar att $(\bar{r}'_\theta \times \bar{r}'_\phi)\cdot\hat{r} > 0$ vilket betyder att $\bar{r}'_\theta \times \bar{r}'_\phi$ pekar ut ur S.

Flödet är

$$\Phi = \iint_S \bar{A}\cdot\hat{n}\,dS = \iint_D \bar{A}\cdot(\bar{r}'_\theta \times \bar{r}'_\phi)\,d\theta\,d\phi \quad \text{där } D:\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$$

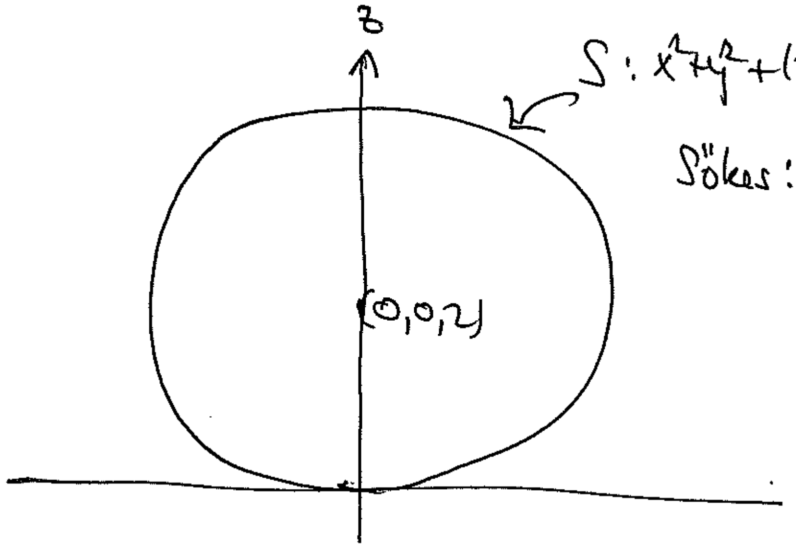
$$= \iint_D (2-\cos\theta)\,\hat{r}\cdot\left(\frac{\sin\theta}{(2-\cos\theta)^2}\,\hat{r} + \frac{\sin^2\theta}{(2-\sin\theta)^3}\,\hat{\theta}\right) d\theta\,d\phi$$

$$= \iint_D \frac{\sin\theta}{2-\cos\theta}\,d\theta\,d\phi$$

$$= \int_0^\pi \left(\int_0^{2\pi} \frac{\sin\theta}{2-\cos\theta}\,d\phi\right) d\theta$$

$$= 2\pi \int_0^\pi \frac{\sin\theta}{2-\cos\theta}\,d\theta = 2\pi \Big[\ln(2-\cos\theta)\Big]_0^\pi = \underline{\underline{2\pi\ln 3}}$$

9,18 (en alternativ lösning)



$S: x^2+y^2+(z-2)^2=4$

Sökes: flödet av $\bar{A} = \dfrac{z(\rho^2-1)}{\rho}\hat{\rho}$ ut genom S.

· Vi parametriserar ytan S med hjälp av sfäriska koordinater: på S är

$$x = 2\sin\theta\cos\phi\,\hat{x} + 2\sin\theta\sin\phi\,\hat{y} + (2+2\cos\theta)\,\hat{z}$$
$$= 2\sin\theta\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right) + 2(1+\cos\theta)\hat{z}$$
$$= 2\sin\theta\,\hat{\rho} + 2(1+\cos\theta)\hat{z}$$

med $(\theta,\phi) \in D$ där $D:\ 0 \le \theta \le \pi,\quad 0 \le \phi \le 2\pi$.

· $$\bar{r}'_\theta \times \bar{r}'_\phi = \left(2\cos\theta\,\hat{\rho} - 2\sin\theta\,\hat{z}\right) \times 2\sin\theta\,\hat{\phi} \quad \left[\text{ty } \frac{\partial}{\partial\phi}\hat{\rho} = \hat{\phi}\right]$$
$$= 4\sin^2\theta(\hat{\phi}\times\hat{z}) + 4\sin\theta\cos\theta(\hat{\rho}\times\hat{\phi})$$
$$= 4\sin^2\theta\,\hat{\rho} + 4\sin\theta\cos\theta\,\hat{z}$$

· enhetsnormalen $\hat{n}$ till S ska peka ut ur S och då måste $\hat{n}$ ha positiv $\hat{\rho}$-komponent, vilket ger $\hat{n}\cdot\hat{\rho} > 0$. Detta ger att

$$\hat{n} = \frac{\bar{r}'_\theta \times \bar{r}'_\phi}{|\bar{r}'_\theta \times \bar{r}'_\phi|}$$

och vi erhåller nu att flödet Φ ges genom

$$\Phi = \iint_S \bar{A}\cdot\hat{n}\,dS = \iint_D \bar{A}\cdot(\bar{r}'_\theta \times \bar{r}'_\phi)\,d\theta\,d\phi$$

På S är $\rho = \sqrt{x^2+y^2} = 2\sin\theta$, $z = 2(1+\cos\theta)$ – se ovan –
och vi har

$$\bar{A} = \frac{2(1+\cos\theta)(4\sin^2\theta - 1)}{2\sin\theta}\,\hat{\rho} \quad \text{på } S.$$

Då är

$$\Phi = \iint_D \frac{2(1+\cos\theta)(4\sin^2\theta-1)}{2\sin\theta}\,\hat{\rho}\cdot\left(4\sin^2\theta\,\hat{\rho} + 4\sin\theta\cos\theta\,\hat{z}\right)d\theta\,d\phi$$

$$= 4\iint_D (1+\cos\theta)(4\sin^2\theta-1)\sin\theta\,d\theta\,d\phi$$

$$= 4\int_0^\pi\left(\int_0^{2\pi}(1+\cos\theta)(4\sin^2\theta-1)\sin\theta\,d\phi\right)d\theta$$

$$= 8\pi\int_0^\pi (1+\cos\theta)(4\sin^2\theta-1)\sin\theta\,d\theta$$

$$= \left[t=\cos\theta;\ dt=-\sin\theta\,d\theta;\ \sin^2\theta = 1-t^2\right]$$

$$= -8\pi\int_1^{-1}(1+t)(3-4t^2)\,dt$$

$$= 8\pi\int_{-1}^1 (1+t)(3-4t^2)\,dt$$

$$= 8\pi\int_{-1}^1 (3-4t^2)\,dt \quad \text{ty} \quad \int_{-1}^1 t(3-4t^2)\,dt = 0$$

$$= 16\pi\int_0^1 (3-4t^2)\,dt$$

$$= \underline{\underline{\frac{80\pi}{3}}}$$

9.20 S parametriseras genom

$$x = \sin\theta\cos\phi\,,\quad y = \sin\theta\sin\phi\,,\quad z = 1+\cos\theta$$

och ortsvektorn för S är då

$$\bar{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + (1+\cos\theta)\hat{z}$$
$$= \sin\theta(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + (1+\cos\theta)\hat{z}$$
$$= \sin\theta\,\hat{\rho} + (1+\cos\theta)\hat{z} \quad \text{med} \quad (\theta,\phi)\in D \quad \text{där}$$

$$D:\ 0\leq\theta\leq\pi$$
$$0\leq\phi\leq 2\pi.$$

$$\bar{r}'_\theta\times\bar{r}'_\phi = (\cos\theta\,\hat{\rho} - \sin\theta\,\hat{z})\times(\sin\theta\,\hat{\phi})$$
$$= \sin^2\theta(\hat{\phi}\times\hat{z}) + \sin\theta\cos\theta(\hat{\rho}\times\hat{\phi})$$
$$= \sin^2\theta\,\hat{\rho} + \sin\theta\cos\theta\,\hat{z}$$

som pekar ut ur S ty $(\bar{r}'_\theta\times\bar{r}'_\phi)\cdot\hat{\rho} > 0$.

Vi har då att flödet är

$$\Phi = \iint_S \bar{A}\cdot\hat{n}\,dS$$
$$= \iint_D \bar{A}\cdot(\bar{r}'_\theta\times\bar{r}'_\phi)\,d\theta\,d\phi$$
$$= \iint_D \hat{\rho}\cdot(\sin^2\theta\,\hat{\rho} + \sin\theta\cos\theta\,\hat{z})\,d\theta\,d\phi$$
$$= \iint_D \sin^2\theta\,d\theta\,d\phi = \int_0^{\pi}\left(\int_0^{2\pi}\sin^2\theta\,d\phi\right)d\theta$$
$$= 2\pi\cdot\int_0^{\pi}\sin^2\theta\,d\theta = \pi\int_0^{\pi}(1-\cos 2\theta)\,d\theta = \underline{\underline{\pi^2}}$$