Lektion 6

2.26 Låt $z = z(u(x,y), v(x,y))$, $x \geq 0, y \geq 0$,

där $\begin{cases} u = xy^2 \\ v = y \end{cases}$ $\Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \underbrace{\frac{\partial u}{\partial x}}_{=y^2} + \frac{\partial z}{\partial v} \underbrace{\frac{\partial v}{\partial x}}_{=0} = y^2 \frac{\partial z}{\partial u},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \underbrace{\frac{\partial u}{\partial y}}_{=2xy} + \frac{\partial z}{\partial v} \underbrace{\frac{\partial v}{\partial y}}_{=1} = 2xy \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Insättning i $2x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = y + xy$ ger

$$\cancel{2xy^2 \frac{\partial z}{\partial u}} - \cancel{2xy^2 \frac{\partial z}{\partial u}} - y \frac{\partial z}{\partial v} = y + xy \Rightarrow$$

$$\frac{\partial z}{\partial v} = -1 - x.$$

Observera att $x = \frac{u}{y^2} = \frac{u}{v^2} \Rightarrow$

$$\boxed{\frac{\partial z}{\partial v} = -1 - \frac{u}{v^2}}$$ är den nya ekvationen.

Vi ser att $z = -\int \left(1 + \frac{u}{v^2}\right) dv + f(u) =$

$$= -v + \frac{u}{v} + f(u),$$

där $f \in C^1$ är godtycklig.

Tillbaka till x och y:

$$z(x,y) = -y + xy + f(xy^2), \quad f \in C^1$$

Antag nu att $z(1,y) = e^{-y}$, $y>0$ $\Rightarrow$ (byt $y^2 \to t$)

$-\cancel{y} + \cancel{y} + f(y^2) = e^{-y} \Leftrightarrow f(y^2) = e^{-\sqrt{y^2}}$ d v s

$f(t) = e^{-\sqrt{t}}$ i detta fall. Lösningen blir

$z(x,y) = -y + xy + e^{-\sqrt{xy^2}} \Rightarrow [y \geq 0]$

$$\underline{z(x,y) = -y + xy + e^{-y\sqrt{x}}}.$$

2.30 Låt $y = y(u(x)) = y(\overbrace{\ln x}^{=u})$.

I så fall $\frac{dy}{dx} = y'(\ln x) \cdot \frac{1}{x}$,

$$\frac{d^2y}{dx^2} = \left(y'(\ln x) \cdot \frac{1}{x}\right)' = y''(\ln x) \cdot \frac{1}{x^2} + y'(\ln x)\left(-\frac{1}{x^2}\right).$$

Insättning i $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 3y = 5x^2$ $x>0$

ger

$$y''(\ln x) - y'(\ln x) + 3y'(\ln x) - 3y(\ln x) = 5x^2.$$

Eftersom $u = \ln x$ och $x = e^u$, blir detta

$$\boxed{y''(u) + 2y'(u) - 3y(u) = 5e^{2u}}$$

$y_h = C_1 e^{u} + C_2 e^{-3u}$ eftersom $\lambda^2 + 2\lambda - 3 = 0$ har rötter $\lambda_1 = 1$ och $\lambda_2 = -3$.

Insättning $y_p(u) = Ce^{2u}$ ger

$4Ce^{2u} + 4Ce^{2u} - 3Ce^{2u} = 5e^{2u} \Rightarrow 5C = 5 \Rightarrow C = 1$

Vi ser att $y_p = e^{2u}$, vilket ger

$$y = y_h + y_p = C_1 e^{u} + C_2 e^{-3u} + e^{2u}.$$

Vi byter tillbaka mot x!

$$y = C_1 e^{\ln x} + C_2 e^{-3\ln x} + e^{2\ln x} \Rightarrow$$

$$\underbrace{y = C_1 x + \frac{C_2}{x^3} + x^2, \quad C_1, C_2 - \text{konst.}}$$

2.31 Låt $z = z(u(x), v(x))$, $\begin{cases} u = \ln x \\ v = x^2 \end{cases}$.

I så fall

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx} = \underbrace{\frac{1}{x}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}}.$$

$$\frac{d^2 z}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right) = |\text{produktregel}| =$$

$$= -\frac{1}{x^2}\cdot\frac{\partial z}{\partial u} + \frac{1}{x}\frac{d}{dx}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial v} + 2x\frac{d}{dx}\left(\frac{\partial z}{\partial v}\right) = \otimes$$

Observera att $\frac{\partial z}{\partial u}$ har samma slags beroende av u, v och x som z, dvs

$$\frac{\partial z}{\partial u} = \left(\frac{\partial z}{\partial u}\right)(u(x), v(x)) \Rightarrow$$

kedjeregeln fungerar på samma sätt som när vi beräknade $\frac{dz}{dx}$!

$$\frac{d}{dx}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial z}{\partial u}\right)'_u \frac{du}{dx} + \left(\frac{\partial z}{\partial u}\right)'_v \frac{dv}{dx} =$$

$$= \frac{\partial^2 z}{\partial u^2}\cdot\frac{1}{x} + \frac{\partial^2 z}{\partial u \partial v}\cdot 2x.$$

Likadant beräknar vi

$$\frac{d}{dx}\left(\frac{\partial z}{\partial v}\right) = \left(\frac{\partial z}{\partial v}\right)'_u \frac{du}{dx} + \left(\frac{\partial z}{\partial v}\right)'_v \frac{dv}{dx} =$$

$$= \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{1}{x} + \frac{\partial^2 z}{\partial v^2} \cdot 2x$$

$\frac{d^2 z}{dx^2}$ blir alltså

$$\frac{d^2 z}{dx^2} = \otimes = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{1}{x} + 2x\cdot\frac{\partial^2 z}{\partial u \partial v}\right] +$$

$$+ 2\frac{\partial z}{\partial v} + 2x\left[\frac{\partial^2 z}{\partial v \partial u}\cdot\frac{1}{x} + 2x\frac{\partial^2 z}{\partial v^2}\right] =$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + 4\frac{\partial^2 z}{\partial u \partial v} +$$

$$+ 4x^2\frac{\partial^2 z}{\partial v^2},$$

där vi har använt faktumet att $z \in C^2 \Rightarrow$ $z''_{uv} = z''_{vu}$.

Svar: $$\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + 4\frac{\partial^2 z}{\partial u \partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}.$$

2.32

Låt $z = z(u(x,y), v(x,y))$ där $\begin{cases} u = 2x + y \\ v = x \end{cases}$.

I så fall $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=2} + \frac{\partial z}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{=1} = 2\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{=1} + \frac{\partial z}{\partial v}\underbrace{\frac{\partial v}{\partial y}}_{=0} = \frac{\partial z}{\partial u},$$

$$\frac{\partial^2 z}{\partial x^2} = \left(2\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)'_x = 2\left(\frac{\partial z}{\partial u}\right)'_x + \left(\frac{\partial z}{\partial v}\right)'_x =$$

$$= 2\left[\left(\frac{\partial z}{\partial u}\right)'_u \underbrace{\frac{\partial u}{\partial x}}_{=2} + \left(\frac{\partial z}{\partial u}\right)'_v \underbrace{\frac{\partial v}{\partial x}}_{=1}\right] +$$

$$+ \left[\left(\frac{\partial z}{\partial v}\right)'_u \underbrace{\frac{\partial u}{\partial x}}_{=2} + \left(\frac{\partial z}{\partial v}\right)'_v \underbrace{\frac{\partial v}{\partial x}}_{=1}\right] =$$

$$= 4z''_{uu} + 2z''_{uv} + 2z''_{vu} + z''_{vv} = \left[\begin{matrix} z \in C^2 \Rightarrow \\ z''_{uv} = z''_{vu} \end{matrix}\right] =$$

$$= 4z''_{uu} + 4z''_{uv} + z''_{vv},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left[z \in C^2\right] = \frac{\partial^2 z}{\partial y \partial x} = \left(\frac{\partial z}{\partial u}\right)'_x =$$

$$= \left(\frac{\partial z}{\partial u}\right)'_u \frac{\partial u}{\partial x} + \left(\frac{\partial z}{\partial u}\right)'_v \frac{\partial v}{\partial x} = 2z''_{uu} + z''_{uv},$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial z}{\partial u}\right)'_y = \left(\frac{\partial z}{\partial u}\right)'_u \underbrace{\frac{\partial u}{\partial y}}_{=1} + \left(\frac{\partial z}{\partial u}\right)'_v \underbrace{\frac{\partial v}{\partial y}}_{=0} =$$

$$= z''_{uu}$$

Insättning i $z''_{xx} - 4z''_{xy} + 4z''_{yy} = 6y$, $z \in C^2$ ger (OBS! $y = u - 2x = u - 2v$)!

$$\cancel{4z''_{uu}} + \cancel{4z''_{uv}} + z''_{vv} - \cancel{8z''_{uu}} - \cancel{4z''_{uv}} + \cancel{4z''_{uu}} = 6u - 12v$$

$$\boxed{z''_{vv} = 6u - 12v}$$ – så här ser ekvationen ut efter variabelbytet!

Vi integrerar! $z'_v = \int (6u - 12v)\,dv + f(u) =$

$$= 6uv - 6v^2 + f(u)$$

så $z = \int \left(6uv - 6v^2 + f(u)\right) dv + g(u) =$

$$= \underline{3uv^2 - 2v^3 + f(u)\cdot v + g(u)}$$

$\begin{cases} u = 2x+y \\ v = x \end{cases}$ ger

$$z = 3(2x+y)x^2 - 2x^3 + f(2x+y)\cdot x + g(2x+y) \iff$$

$$\boxed{z = 4x^3 + 3x^2y + xf(2x+y) + g(2x+y)} \quad (*)$$

$g, f \in C^2$ är godtyckliga.

b) Antag att $z(0,y) = e^{-y^2} \Rightarrow$ från $(*)$ viseratt

$$e^{-y^2} = g(y) \Rightarrow g(2x+y) = e^{-(2x+y)^2}.$$

I så fall, $\boxed{z = 4x^3 + 3x^2y + xf(2x+y) + e^{-(2x+y)^2}}$ $(\star)$

$f \in C^2$ är godtycklig.

c) Antag dessutom att $z'_x(0,y) = 0$. Eftersom

från $(\star)$

$$z'_x = 12x^2 + 6xy + f(2x+y) + x\cdot f'(2x+y)\cdot 2 + e^{-(2x+y)^2}\left(-2(2x+y)\right)\cdot 2 =$$

$$= 12x^2 + 6xy + f(2x+y) + 2xf'(2x+y) - 4(2x+y)e^{-(2x+y)^2},$$

$$\Rightarrow 0 = f(y) - 4ye^{-y^2} \iff f(y) = 4ye^{-y^2} \Rightarrow$$

$$\underline{f(2x+y) = 4(2x+y)e^{-(2x+y)^2}.}$$

I så fall,

$$\boxed{z = 4x^3 + 3x^2y + 4x(2x+y)e^{-(2x+y)^2} + e^{-(2x+y)^2}.}$$

2.33 Låt $f = f(u(x,y), v(x,y))$ med $\begin{cases} u = x+y \\ v = xy \end{cases}$.

Deriverar:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial f}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{y} = \frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \left(\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\right)'_y = \left(\frac{\partial f}{\partial u}\right)'_y + \frac{\partial f}{\partial v} + y\left(\frac{\partial f}{\partial v}\right)'_y =$$

$$= \left(\frac{\partial f}{\partial u}\right)'_u \frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial u}\right)'_v \frac{\partial v}{\partial y} + \frac{\partial f}{\partial v} + y\left[\left(\frac{\partial f}{\partial v}\right)'_u \frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)'_v \frac{\partial v}{\partial y}\right]$$

$$= f''_{uu}\underbrace{\frac{\partial u}{\partial y}}_{=1} + f''_{uv}\underbrace{\frac{\partial v}{\partial y}}_{=x} + \frac{\partial f}{\partial v} + y\left[f''_{vu}\underbrace{\frac{\partial u}{\partial y}}_{=1} + f''_{vv}\underbrace{\frac{\partial v}{\partial y}}_{=x}\right] =$$

$$= f''_{uu} + x f''_{uv} + f'_v + y f''_{vu} + yx f''_{vv} =$$

$$= f''_{uu} + \underbrace{(x+y)}_{=u} f''_{uv} + \underbrace{yx}_{=v} f''_{vv} + f'_v =$$

$$= f''_{uu} + u f''_{uv} + v f''_{vv} + f'_v$$

där vi har använt $z \in C^2 \Rightarrow f''_{uv} = f''_{vu}$.

2.34 Låt $z = z(u(x,y), v(x,y))$, där $\begin{cases} u = 2xy \\ v = \frac{1}{y} \end{cases}$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\overbrace{\frac{\partial u}{\partial x}}^{=2y} + \frac{\partial z}{\partial v}\overbrace{\frac{\partial v}{\partial x}}^{=0} = 2y\frac{\partial z}{\partial u},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(2y \frac{\partial z}{\partial u}\right)'_y = 2\frac{\partial z}{\partial u} + 2y\left(\frac{\partial z}{\partial u}\right)'_y =$$

$$= 2z'_u + 2y\left[\left(\frac{\partial z}{\partial u}\right)'_u \frac{\partial u}{\partial y} + \left(\frac{\partial z}{\partial u}\right)'_v \frac{\partial v}{\partial y}\right] =$$

$$= 2z'_u + 2y\left[z''_{uu} \cdot 2x + z''_{uv}\left(-\frac{1}{y^2}\right)\right] =$$

$$= 2z'_u + 4xy \cdot z''_{uu} - \frac{2}{y} z''_{uv} .$$

$$\frac{\partial^2 z}{\partial x^2} = \left(2y\frac{\partial z}{\partial u}\right)'_x = 2y\left(\frac{\partial z}{\partial u}\right)'_x =$$

$$= 2y\left[\left(\frac{\partial z}{\partial u}\right)'_u \frac{\partial u}{\partial x} + \left(\frac{\partial z}{\partial u}\right)'_v \frac{\partial v}{\partial x}\right] =$$

$$= 2y\left[z''_{uu} \cdot 2y + 0\right] = 4y^2 z''_{uu} .$$

Insättning i $x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = x$ ger

$$\cancel{4y^2x \cdot z''_{uu}} - \cancel{2yz'_u} - \cancel{4xy^2 z''_{uu}} + 2z''_{uv} + \cancel{2y \cdot z'_u} = x$$

$\Rightarrow 2z''_{uv} = x$, där x kan skrivas $x = \frac{uv}{2}$ $\Rightarrow$

ekvationen blir $\boxed{z''_{uv} = \frac{uv}{4}}$

Integration ger:

$$z'_u = \int \frac{uv}{4}\, dv + f(u) = \frac{uv^2}{8} + f(u) \Rightarrow$$

$$z = \int\left(\frac{uv^2}{8} + f(u)\right) du + g(v) = \frac{u^2v^2}{16} + \underbrace{\int f(u)\, du}_{=h(u)} + g(v)$$

$$z = \frac{u^2v^2}{16} + h(u) + g(v) , \quad h, g \in C^2$$

Tillbaka till $\begin{matrix} u = 2xy \\ v = \frac{1}{y} \end{matrix}$:

$$z = \frac{4x^2y^2}{16y^2} + \underbrace{h(2xy)}_{=k(xy)} + \underbrace{g\left(\frac{1}{y}\right)}_{=\ell(y)} \Rightarrow$$

$$\boxed{z = \frac{x^2}{4} + k(xy) + \ell(y)}$$ där k och ℓ är godtyckliga funktioner i klass C^2.

2.38

a) Låt $u = u(x(\rho,\varphi), y(\rho,\varphi))$ där $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$ (*)

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} = \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\cdot\sin\varphi \Rightarrow$$

$$\boxed{u'_\rho = u'_x\cdot\cos\varphi + u'_y\cdot\sin\varphi}$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} = -u'_x\rho\sin\varphi + u'_y\cdot\rho\cos\varphi$$

$$\boxed{u'_\varphi = -\rho\sin\varphi\cdot u'_x + \rho\cos\varphi\cdot u'_y}$$

b) $$\begin{cases} u'_\rho = u'_x\cdot\cos\varphi + u'_y\cdot\sin\varphi & (1) \\ u'_\varphi = -\rho\sin\varphi\, u'_x + \rho\cos\varphi\cdot u'_y & (2) \end{cases}$$

löser systemet för u'_x och u'_y !

$\rho \cdot \sin\varphi \cdot (1) + \cos\varphi \cdot (2)$ ger

$$\rho \sin\varphi\, u'_\rho + \cos\varphi \cdot u'_\varphi = u'_x \left(\cancel{\rho \sin\varphi \cos\varphi} - \cancel{\rho \cos\varphi \sin\varphi} \right) + u'_y \underbrace{\left(\rho \sin^2\varphi + \rho \cos^2\varphi \right)}_{=\rho} \Longrightarrow$$

$$\boxed{u'_y = \sin\varphi \cdot u'_\rho + \frac{\cos\varphi}{\rho} \cdot u'_\varphi} \qquad (3)$$

$\rho \cos\varphi \cdot (1) - \sin\varphi \cdot (2)$ ger

$$\rho \cos\varphi \cdot u'_\rho - \sin\varphi \cdot u'_\varphi = u'_x \overbrace{\left(\rho \cos^2\varphi + \rho \sin^2\varphi \right)}^{=\rho} + u'_y \left(\cancel{\rho \cos\varphi \sin\varphi} - \cancel{\rho \cos\varphi \sin\varphi} \right) \Rightarrow$$

$$\boxed{u'_x = \cos\varphi \cdot u'_\rho - \frac{\sin\varphi}{\rho} \cdot u'_\varphi} \qquad (4)$$

c) Vi behöver $\frac{\partial^2 u}{\partial x^2}$ och $\frac{\partial^2 u}{\partial y^2}$:

$$\frac{\partial^2 u}{\partial x^2} = \left(\cos\varphi \cdot u'_\rho - \frac{\sin\varphi}{\rho} \cdot u'_\varphi \right)'_x =$$

$$= \underbrace{(\cos\varphi)'_x\, u'_\rho}_{1} + \underbrace{\cos\varphi \cdot (u'_\rho)'_x}_{2} - \underbrace{\left(\frac{\sin\varphi}{\rho} \right)'_x \cdot u'_\varphi}_{3} - \underbrace{\frac{\sin\varphi}{\rho} \cdot (u'_\varphi)'_x}_{4} = \otimes$$

Vi ska beräkna 1 – 4 separat. Observera att $\rho = \sqrt{x^2+y^2}$, $\cos\varphi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 u}{\partial x^2} = (u'_x)'_x = \left[\begin{array}{c}\text{använd (4) men byt}\\ u \text{ mot } u'_x\,!\end{array}\right] =$$

$$= \cos\varphi\,(u'_x)'_\rho - \frac{\sin\varphi}{\rho}\cdot(u'_x)'_\varphi = \left[\begin{array}{c}\text{använd}\\ \text{(4) igen}\end{array}\right] =$$

$$= \cos\varphi\left(\cos\varphi\cdot u'_\rho - \frac{\sin\varphi}{\rho}u'_\varphi\right)'_\rho -$$

$$- \frac{\sin\varphi}{\rho}\left(\cos\varphi\cdot u'_\rho - \frac{\sin\varphi}{\rho}u'_\varphi\right)'_\varphi =$$

$$= \cos\varphi\left[\cos\varphi\cdot u''_{\rho\rho} + \frac{\sin\varphi}{\rho^2}u'_\varphi - \frac{\sin\varphi}{\rho}\cdot u''_{\varphi\rho}\right]$$

$$- \frac{\sin\varphi}{\rho}\left[-\sin\varphi\cdot u'_\rho + \cos\varphi\cdot u''_{\rho\varphi} - \frac{\cos\varphi}{\rho}u'_\varphi - \frac{\sin\varphi}{\rho}u''_{\varphi\varphi}\right] =$$

$$= \cos^2\varphi\cdot u''_{\rho\rho} + \frac{\cos\varphi\sin\varphi}{\rho^2}u'_\varphi - \frac{\cos\varphi\sin\varphi}{\rho}u''_{\varphi\rho}$$

$$+ \frac{\sin^2\varphi}{\rho}u'_\rho - \frac{\sin\varphi\cos\varphi}{\rho}u''_{\rho\varphi} + \frac{\sin\varphi\cos\varphi}{\rho^2}u'_\varphi + \frac{\sin^2\varphi}{\rho^2}u''_{\varphi\varphi}.$$

$$= \cos^2\varphi\cdot u''_{\rho\rho} - \frac{2\sin\varphi\cos\varphi}{\rho}u''_{\rho\varphi} + \frac{\sin^2\varphi}{\rho^2}u''_{\varphi\varphi}$$

$$+ \frac{\sin^2\varphi}{\rho}u'_\rho + \frac{2\sin\varphi\cos\varphi}{\rho^2}u'_\varphi$$

Det som är viktigt här är att vi kan använda (3) och (4) för olika funktioner – dem behöver inte vara u! Kan vara t ex u'_x eller u'_y, eftersom t ex

$$u'_x = (u'_x)\big(\rho(x,y),\ \varphi(x,y)\big), \quad \text{exakt som}$$

$$u = u\big(\rho(x,y), \varphi(x,y)\big).$$ (Beror på samma variabler!)

Vi beräknar $\frac{\partial^2 u}{\partial y^2}$ på liknande sätt

$$\frac{\partial^2 u}{\partial y^2} = \left(\sin\varphi \cdot u'_\rho + \frac{\cos\varphi}{\rho} \cdot u'_\varphi\right)'_y =$$

$$= \sin\varphi \left[\sin\varphi \cdot u'_\rho + \frac{\cos\varphi}{\rho} \cdot u'_\varphi\right]'_\rho +$$

använd (3) med $\sin\varphi \cdot u'_\rho + \frac{\cos\varphi}{\rho} u'_\varphi$ istället för u!

$$+ \frac{\cos\varphi}{\rho}\left[\sin\varphi \cdot u'_\rho + \frac{\cos\varphi}{\rho} \cdot u'_\varphi\right]'_\varphi =$$

$$= \sin\varphi\left[\sin\varphi \cdot u''_{\rho\rho} - \frac{\cos\varphi}{\rho^2} \cdot u'_\varphi + \frac{\cos\varphi}{\rho} u''_{\varphi\rho}\right]$$

$$+ \frac{\cos\varphi}{\rho}\left[\cos\varphi \cdot u'_\rho + \sin\varphi \cdot u''_{\rho\varphi} - \frac{\sin\varphi}{\rho} u'_\varphi + \frac{\cos\varphi}{\rho} u''_{\varphi\varphi}\right] =$$

$$= \sin^2\varphi \cdot u''_{\rho\rho} - \frac{\sin\varphi\cos\varphi}{\rho^2} u'_\varphi + \frac{\sin\varphi\cos\varphi}{\rho} u''_{\rho\varphi}$$

$$+ \frac{\cos^2\varphi}{\rho} u'_\rho + \frac{\cos\varphi\sin\varphi}{\rho} u''_{\rho\varphi} - \frac{\cos\varphi\sin\varphi}{\rho^2} u'_\varphi + \frac{\cos^2\varphi}{\rho^2} u''_{\varphi\varphi}$$

$$= \sin^2\varphi \cdot u''_{\rho\rho} + \frac{2\sin\varphi\cos\varphi}{\rho} u''_{\rho\varphi} + \frac{\cos^2\varphi}{\rho^2} u''_{\varphi\varphi}$$

$$+ \frac{\cos^2\varphi}{\rho} u'_\rho - \frac{2\cos\varphi\sin\varphi}{\rho^2} u'_\varphi . \quad \Rightarrow$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u''_{\rho\rho} + \frac{1}{\rho^2} u''_{\varphi\varphi} + \frac{1}{\rho} u'_\rho ,}$$

där vi har förkortat några termer och använt $\cos^2\varphi + \sin^2\varphi = 1$. Ekvationen blir

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

2.36 c

Vi har sett i (2.38) att i polära koordinater Laplaces ekvation kan skrivas som

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0 \qquad (*)$$

Antag att u beror inte på φ $\Rightarrow$ $\frac{\partial^2 u}{\partial \varphi^2} = 0$ $\Rightarrow$

$(*)$ blir $\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} = 0$ eller

$$u''(\rho) + \frac{1}{\rho}u'(\rho) = 0. \qquad (\star)$$

Låt $v(\rho) = u'(\rho) \Rightarrow u''(\rho) = v'(\rho)$

och $(\star)$ blir

$$v'(\rho) + \frac{1}{\rho}v(\rho) = 0 \quad \text{– linjär ekvation.}$$

$I.F. = e^{\int \frac{1}{\rho}d\rho} = e^{\ln \rho} = \rho$ $\Rightarrow$ multiplicerar ekvationen med ρ:

$$\rho \cdot v'(\rho) + v(\rho) = 0 \Leftrightarrow$$

$$(v(\rho)\cdot\rho)' = 0 \Leftrightarrow$$

$$v(\rho)\cdot\rho = C_1 = \text{konst}$$

$$v(\rho) = \frac{C_1}{\rho} \Rightarrow u'(\rho) = \frac{C_1}{\rho} \Rightarrow$$

$$u(\rho) = C_1 \ln\rho + C_2$$

$$\Rightarrow u(x,y) = C_1 \ln\sqrt{x^2+y^2} + C_2 = \frac{C_1}{2}\ln(x^2+y^2) + C_2.$$

= C – ny konstant

Svar: $u = c_1 \cdot \ln(x^2+y^2) + C_2$.

Extra

2.27 Låt $f = f(u(x,y), v(x,y))$ där $u = \frac{y}{x}$, och v kommer att bestämmas senare.

$$\frac{\partial f}{\partial x} = f'_u \cdot u'_x + f'_v \cdot v'_x = f'_u \cdot \left(-\frac{y}{x^2}\right) + f'_v \cdot v'_x.$$

$$\frac{\partial f}{\partial y} = f'_u \cdot u'_y + f'_v \cdot v'_y = \frac{1}{x} \cdot f'_u + f'_v \cdot v'_y$$

Insättning i $x f'_x(x,y) + y f'_y(x,y) = -f(x,y)$ ger

$$\cancel{-\frac{y}{x} \cdot f'_u} + x \cdot f'_v \cdot v'_x + \cancel{\frac{y}{x} f'_u} + y f'_v \cdot v'_y = -f$$

Låt $v = x \Rightarrow$ ekvationen blir

$$x \cdot f'_v = -f \iff v \cdot f'_v = -f \iff$$

$$\frac{1}{f} f'_v = -\frac{1}{v}$$

$$(\ln|f|)'_v = -\frac{1}{v}$$

$$\ln|f| = -\ln|v| + g(u)$$

$$|f| = e^{-\ln|v| + g(u)} \Rightarrow$$

$$f = \pm e^{g(u)} \cdot \frac{1}{|v|}$$

D v s $f(u,v) = \dfrac{h(u)}{v}$, där $h \in C^1$ är godtycklig.

Vi byter $u = \frac{y}{x}$, $v = x$ $\Rightarrow$

$$f(x,y) = \frac{h\left(\frac{y}{x}\right)}{x}, \quad h \in C^1.$$

<u>2.28</u> Låt $g(t) = f(u(t), v(t))$ där $\begin{cases} u = 2t \\ v = t \end{cases}$

$h(t) = f(u(t), v(t))$ där $\begin{cases} u = t \\ v = -t \end{cases}$

Vad är $g'(t)$ och $h'(t)$ då?

$$\left.\begin{aligned} g'(t) &= f'_u u'_t + f'_v \cdot v'_t = 2f'_u + f'_v \\ h'(t) &= f'_u u'_t + f'_v \cdot v'_t = f'_u - f'_v \end{aligned}\right\} \quad (*)$$

Här u och v är inte längre kopplad till definitionerna ovanför, de representerar bara det första och det andra argumentet.

(Man kan skriva $f'_u = f'_1$ – första argument, $f'_v = f'_2$ – andra argument)

Om $t = 0 \Rightarrow \begin{cases} u = 0 \\ v = 0 \end{cases}$ för både g och h.

Insättning i (*) ger

$$g'(0) = 2f'_1(0,0) + f'_2(0,0)$$
$$h'(0) = f'_1(0,0) - f'_2(0,0)$$

$$
\begin{array}{rl}
 & a = 2f_1'(0,0) + f_2'(0,0) \\
+ & b = f_1'(0,0) - f_2'(0,0) \\
\hline
\end{array}
$$

$$a+b = 3f_1'(0,0) \Rightarrow f_1'(0,0) = \frac{a+b}{3}.$$

Från den första ekvationen

$$f_2'(0,0) = a - 2f_1'(0,0) = a - \frac{2a}{3} - \frac{2b}{3} = \frac{a}{3} - \frac{2b}{3},$$

$$\Rightarrow f_2'(0,0) = \frac{a}{3} - \frac{2b}{3}$$

Svar: $f_1'(0,0) = \dfrac{a+b}{3}$

$f_2'(0,0) = \dfrac{a-2b}{3}.$