Lektion 16

6.1

a) Eftersom

$$\frac{1}{7} = \frac{1}{3+2^2-0} \leq \frac{1}{3+x^2-y} \leq \frac{1}{3+0^2-2} = 1$$

då $0 \leq x \leq 2$, $0 \leq y \leq 2$ ser vi att

$$\iint_D \frac{1}{7}\,dx\,dy \leq \iint_D \frac{dx\,dy}{3+x^2-y^2} \leq \iint_D 1\,dx\,dy$$

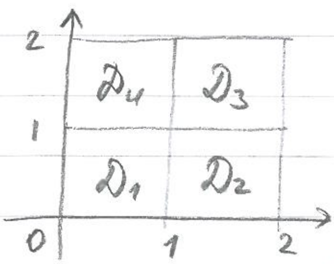
Men $\iint_D 1\,dx\,dy = \text{Area}(D) = 4$

$$\iint_D \frac{1}{7}\,dx\,dy = \frac{1}{7}\iint_D dx\,dy = \frac{1}{7}\text{Area}(D) = \frac{4}{7},$$

så

$$\frac{4}{7} \leq \iint_D \frac{dx\,dy}{3+x^2-y^2} \leq 4.$$

b) Vi delar D i fyra lika stora kvadrater och vi uppskattar f i varje kvadrat uppifrån och underifrån.



(I) $D_1 = \{0 \leq x \leq 1,\ 0 \leq y \leq 1\} \Rightarrow$

$$\frac{1}{4} = \frac{1}{3+1-0} \leq \frac{1}{3+x^2-y} \leq \frac{1}{3+0^2-1} = \frac{1}{2}$$

$$\iint_{D_1} \frac{1}{4}\,dx\,dy \leq \iint_{D_1} \frac{dx\,dy}{3+x^2-y} \leq \iint_{D_1} \frac{1}{2}\,dx\,dy$$

Eftersom $\iint_{D_1} 1\,dx\,dy = \text{Area}(D_1) = 1$ ser vi att

$$\frac{1}{4} \le \iint_{D_1} \frac{dx\,dy}{3+x^2-y} \le \frac{1}{2}$$

(II) $D_2 = \{1 \le x \le 2,\quad 0 \le y \le 1\} \Rightarrow$

$$\frac{1}{7} = \frac{1}{3+2^2-0} \le \frac{1}{3+x^2-y} \le \frac{1}{3+1-1} = \frac{1}{3}$$

$$\iint_{D_2} \frac{1}{7}\,dx\,dy \le \iint_{D_2} \frac{dx\,dy}{3+x^2-y^2} \le \iint_{D_2} \frac{1}{3}\,dx\,dy$$

Eftersom $\iint_{D_2} dx\,dy = \text{Area}(D_2) = 1 \Rightarrow$

$$\frac{1}{7} \le \iint_{D_2} \frac{dx\,dy}{3+x^2-y^2} \le \frac{1}{3}$$

(III) $D_3 = \{1 \le x \le 2,\quad 1 \le y \le 2\} \Rightarrow$

$$\frac{1}{6} = \frac{1}{3+2^2-1} \le \frac{1}{3+x^2-y} \le \frac{1}{3+1-2} = \frac{1}{2}$$

Som i (I) och (II) får vi att

$$\frac{1}{6} \le \iint_{D_3} \frac{dx\,dy}{3+x^2-y} \le \frac{1}{2}$$

då area$(D_3) = 1$.

(IV) $\mathcal{D}_4 = \{ 0 \le x \le 1, \quad 1 \le y \le 2 \} \Rightarrow$

$$\frac{1}{3} = \frac{1}{3+1-1} \le \underline{\frac{1}{3+x^2-y}} \le \frac{1}{3+0-2} = 1$$

Vi ser att

$$\frac{1}{3} \le \iint \frac{dx\,dy}{3+x^2-y} \le 1$$

då area$(\mathcal{D}_4) = 1$.

Nu kan vi uppskatta integralen. Eftersom

$$\iint_{\mathcal{D}} \frac{dx\,dy}{3+x^2-y} = \iint_{\mathcal{D}_1} \cdots + \iint_{\mathcal{D}_2} \cdots + \iint_{\mathcal{D}_3} \cdots + \iint_{\mathcal{D}_4} \cdots ,$$

och vi har uppskattningarna för varje integral i högerledet, ser vi att

$$\frac{1}{4} + \frac{1}{7} + \underbrace{\frac{1}{6} + \frac{1}{3}}_{= \frac{1}{2}} \le \iint_{\mathcal{D}} \frac{dx\,dy}{3+x^2-y} \le \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + 1 \Rightarrow$$

$$\boxed{\frac{25}{28} \le \iint_{\mathcal{D}} \frac{dx\,dy}{3+x^2-y} \le \frac{7}{3}}$$

6.2

a) $\iint_{\mathcal{D}} (x+y)^2\,dx\,dy = \left[\begin{array}{l} \mathcal{D} = \{ (x,y) \in \mathbb{R}^2 ;\ |x| \le 1,\ |y| \le 1 \} = \\ = \{ (x,y) \in \mathbb{R}^2 :\ -1 \le x \le 1,\ -1 \le y \le 1 \} \end{array} \right] =$

$$= \int_{-1}^{1} \left(\int_{-1}^{1} (x+y)^2\,dy \right) dx = \int_{-1}^{1} \left(\int_{-1}^{1} (x^2 + 2xy + y^2)\,dy \right) dx = \boxed{3}$$

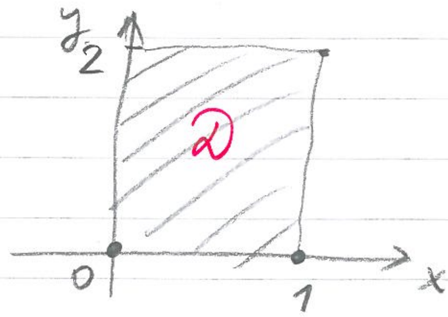
$$= \int_{-1}^{1} \left[x^2y + xy^2 + \frac{y^3}{3} \right]_{y=-1}^{y=1} dx =$$

$$= \int_{-1}^{1} \left(\left(x^2 + \cancel{x} + \frac{1}{3}\right) - \left(-x^2 + \cancel{x} - \frac{1}{3}\right) \right) dx =$$

$$= \int_{-1}^{1} \left(2x^2 + \frac{2}{3}\right) dx = \left[\frac{2x^3}{3} + \frac{2}{3}x \right]_{x=-1}^{x=1} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} =$$

$$= \underline{\frac{8}{3}}$$

b) $$\iint_{\mathcal{D}} \frac{dx\,dy}{1+x+y} =$$



$$\mathcal{D} = \{ 0 \le x \le 1;\ 0 \le y \le 2 \}$$

$$= \int_0^1 \left(\int_0^2 \frac{dy}{1+x+y} \right) dx =$$

$$= \int_0^1 \left[\ln|1+x+y| \right]_{y=0}^{y=2} dx =$$

$$= \int_0^1 \left(\ln|3+x| - \ln|1+x| \right) dx =$$

$$= \int_0^1 \left(\ln(3+x) - \ln(1+x) \right) dx = \otimes$$

Vi behöver $\int \ln t \, dt = \int \ln t \cdot 1 \, dt =$

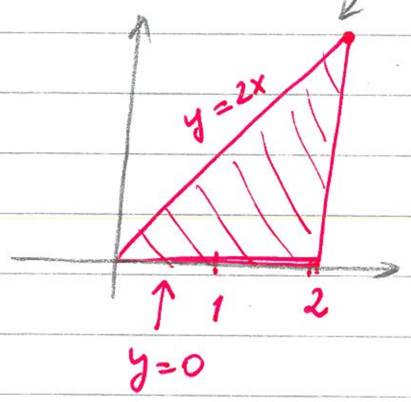
$$= t \ln t - \int t \cdot \frac{1}{t} \, dt = t \ln t - t + C \Rightarrow$$

$$\otimes = \left[(3+x)\ln(3+x) - (3+x) - (1+x)\ln(1+x) + (1+x) \right]_{x=0}^{x=1}$$

$$= \underbrace{4 \ln 4}_{=8\ln 2} - \cancel{4} - 2\ln 2 + \cancel{2} - 3\ln 3 + \cancel{3} - \cancel{1} =$$

$$= \underline{6 \ln 2 - 3 \ln 3}.$$

6.5 a) $0 \le y \le 2x$ är området mellan x-axeln och linjen $y = 2x$. $2x \le 4$ ger restriktionen $x \le 2 \Rightarrow$ området ser ut så här



och kan skrivas som

$$D = \{0 \le x \le 2;\ 0 \le y \le 2x\}$$

vilket ger oss

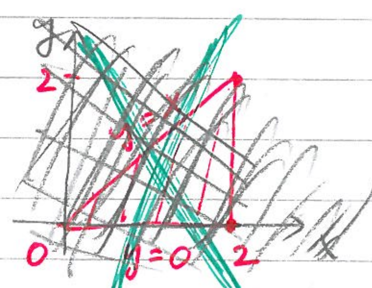
$$\iint_D (xy + y^2)\,dx\,dy =$$

$$= \int_0^2 \left(\int_0^{2x} (xy + y^2)\,dy \right) dx =$$

$$= \int_0^2 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2x} dx = \int_0^2 \left(\frac{4x^3}{2} + \frac{8x^3}{3} \right) dx =$$

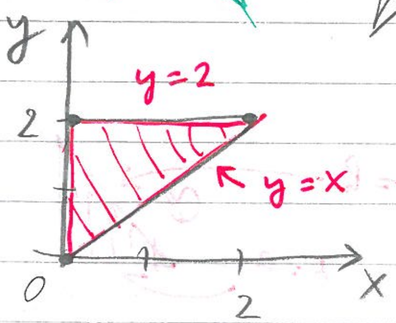
$$= \left[\frac{x^4}{2} + \frac{2x^4}{3} \right]_{x=0}^{x=2} = 8 + \frac{32}{3} = \underline{\underline{\frac{56}{3}}}$$

b)



$$D = \{0 \le x \le 2,\ x \le y \le 2\}$$

vilket ger oss



$$\iint_D (x-y)e^{x+y}\,dx\,dy =$$

$$= \int_0^2 \left(\int_x^2 (x-y)e^{x+y}\,dy \right) dx =$$

$$= \int_0^2 \left(\int_x^2 xe^x \cdot e^y - e^x \cdot ye^y \right) dy \Big) dx$$

konstanter map y

$$= \int_0^2 \left(xe^x \int_x^2 e^y\,dy - e^x \int_x^2 ye^y\,dy \right) dx = \otimes$$

Vi behöver

$$\int y e^y \, dy = ye^y - \int e^y \, dy = ye^y - e^y \Rightarrow$$

$$\otimes = \int_0^2 \left(xe^x \left[e^y \right]_{y=x}^{y=2} - e^x \left[ye^y - e^y \right]_{y=x}^{y=2} \right) dx =$$

$$= \int_0^2 \left(xe^x(e^2 - e^x) - e^x(2e^2 - e^2 - xe^x + e^x) \right) dx =$$

$$= \int_0^2 \left(xe^{2+x} - \cancel{xe^{2x}} - 2e^{2+x} + e^{2+x} + \cancel{xe^{2x}} - e^{2x} \right) dx$$

$$= \int_0^2 \left(xe^{2+x} - e^{2+x} - e^{2x} \right) dx =$$

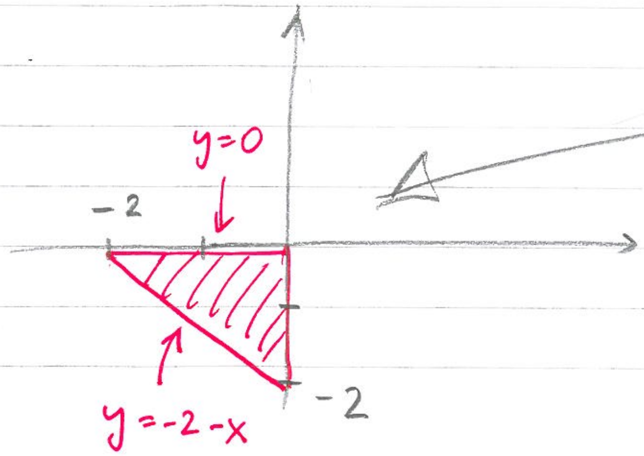
$$= e^2 \left[xe^x - e^x \right]_{x=0}^{x=2} - \left[e^{2+x} \right]_{x=0}^{x=2} - \left[\frac{e^{2x}}{2} \right]_{x=0}^{x=2} =$$

$$= e^2(2e^2 - e^2 - 0 + 1) - (e^4 - e^2) - \left(\frac{e^4}{2} - \frac{1}{2} \right) =$$

$$= \cancel{2e^4} - \cancel{e^4} + e^2 - \cancel{e^4} + e^2 - \frac{e^4}{2} + \frac{1}{2} =$$

$$= -\frac{e^4}{2} + 2e^2 + \frac{1}{2} = \underline{\underline{-\frac{1}{2}\left(e^4 - 4e^2 - 1 \right)}}$$

e) $x + y \geq -2 \Leftrightarrow y \geq -2 - x$. Området ser ut så här, och kan skrivas som



$$D = \{-2 \leq x \leq 0, \ -2 - x \leq y \leq 0\},$$

vilket ger oss integralen.

$$\iint_{\mathcal{D}} (2+x+y)\,dx\,dy = \int_{-2}^{0} \left(\int_{-2-x}^{0} (2+x+y)\,dy \right) dx =$$

$$= \int_{-2}^{0} \left[2y + xy + \frac{y^2}{2} \right]_{y=-2-x}^{y=0} dx =$$

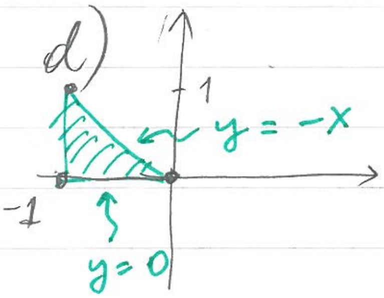
$$= \int_{-2}^{0} \left(0+0+0 - 2(-2-x) - x(-2-x) - \frac{(-2-x)^2}{2} \right) dx =$$

$$= \int_{-2}^{0} \left(4 + 2x + \cancel{2x} + x^2 - 2 - \cancel{2x} - \frac{x^2}{2} \right) dx =$$

$$= \int_{-2}^{0} \left(\frac{x^2}{2} + 2x + 2 \right) dx = \left[\frac{x^3}{6} + x^2 + 2x \right]_{x=-2}^{x=0} =$$

$$= 0+0+0 - \left(\frac{-8}{6} + 4 - 4 \right) = \frac{8}{6} = \frac{4}{3}$$

d)



$$\mathcal{D} = \{-1 \le x \le 0;\ 0 \le y \le -x\} \Rightarrow$$

$$\iint_{\mathcal{D}} e^{x^2}\,dx\,dy =$$

$$= \int_{-1}^{0} \left(\int_{0}^{-x} e^{x^2}\,dy \right) dx =$$

$$= \int_{-1}^{0} \left[e^{x^2} \cdot y \right]_{y=0}^{y=-x} dx = \int_{-1}^{0} \left(-x e^{x^2} - 0 \right) dx =$$

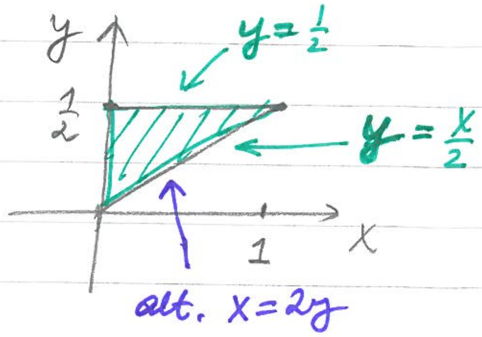
$$= -\int_{-1}^{0} x e^{x^2}\,dx = \left[\begin{array}{ll} x^2 = t & -1 \le x \le 0 \\ dt = 2x\,dx & 1 \ge t \ge 0 \end{array} \right] =$$

$$= -\int_1^0 e^t \frac{dt}{2} = -\frac{1}{2}\left[e^t\right]_{t=1}^{t=0} = -\frac{1}{2}(1-e) =$$

$$= \frac{e-1}{2}.$$

e) $0 \le x \le 2y \le 1 \iff$ delar allt med 2

$0 \le \frac{x}{2} \le y \le \frac{1}{2}$



Vi kan skriva

$$D = \left\{0 \le x \le 1,\ \frac{x}{2} \le y \le \frac{1}{2}\right\} \text{ men då}$$

$$\iint_D \frac{x^3}{1+y^5}\,dx\,dy = \int_0^1 \left(\int_{\frac{x}{2}}^{\frac{1}{2}} \frac{x^3}{1+y^5}\,dy\right)dx$$

och den innre integralen verkar vara svår att beräkna.

I sådana fall kan det vara bra att byta integrations ordning. Vi skriver istället

$$D = \left\{0 \le y \le \frac{1}{2},\ 0 \le x \le 2y\right\}, \text{ så att}$$

$$\iint_D \frac{x^3}{1+y^5}\,dx\,dy = \int_0^{\frac{1}{2}} \left(\int_0^{2y} \frac{x^3}{\underbrace{1+y^5}_{\text{konst map } x}}\,dx\right)dy =$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^4}{4(1+y^5)}\right]_{x=0}^{x=2y} dy =$$

$$= \int_0^{\frac{1}{2}} \frac{4\,\cancel{16}y^4}{\cancel{4}(1+y^5)}\,dy = \int_0^{\frac{1}{2}} \frac{4\cdot 5y^4}{5(1+y^5)}\,dy =$$

$$= \begin{bmatrix} y^5 = t & 0 \le y \le \frac{1}{2} \\ dt = 5y^4 dy & 0 \le t \le \frac{1}{32} \end{bmatrix} =$$

$$= \int_0^{\frac{1}{32}} \frac{4\,dt}{5(1+t)} = \left[\frac{4}{5} \ln |1+t| \right]_{t=0}^{t=\frac{1}{32}} =$$

$$= \frac{4}{5} \ln \frac{33}{32}$$