Lektion 7

P7 3

a) Observera att $\sqrt{9+x^2} = 3\sqrt{1+\frac{x^2}{9}}$, och vi kan använda den standarta utvecklingen

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \binom{1/2}{2}x^2 + O(x^3) =$$
$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + O(x^3) =$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$$

för att beräkna

$$\sqrt{9+x^2} = 3\sqrt{1+\frac{x^2}{9}} = 3\left(1+\frac{1}{2}\cdot\frac{x^2}{9} - \frac{1}{8}\left(\frac{x^2}{9}\right)^2 + O(x^6)\right)$$
$$= 3 + \frac{3x^2}{18} - \frac{3\cdot x^4}{8\cdot 81} + O(x^6) =$$
$$= 3 + \frac{x^2}{6} - \frac{x^4}{216} + O(x^6).$$

b) Observera att $\ln(2-x) = \ln(2(1-\frac{x}{2})) =$

$= \ln 2 + \ln(1-\frac{x}{2})$.

Vi använder $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$ och beräknar

$$\ln(2-x) = \ln 2 + \ln(1-\frac{x}{2}) = \left[\begin{array}{l}\text{sätt } -\frac{x}{2} \text{ istället} \\ \text{för } x \text{ i } \ln(1+x)\end{array}\right]$$
$$= \ln 2 + \left(-\frac{x}{2}\right) - \frac{\left(-\frac{x}{2}\right)^2}{2} + \frac{\left(-\frac{x}{2}\right)^3}{3} - \frac{\left(-\frac{x}{2}\right)^4}{4} + O(x^5) =$$

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$$= \ln 2 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} + O(x^5)$$

c) $\cos\left(2x + \frac{\pi}{2}\right) = -\sin 2x$, och

$\sin x = x - \frac{x^3}{3!} + O(x^5) \Rightarrow$

$$\cos\left(2x + \frac{\pi}{2}\right) = -\sin 2x =$$

$$= -\left[2x - \frac{(2x)^3}{3!} + O(x^5)\right] =$$

$$= -2x + \frac{8x^3}{6} + O(x^5) =$$

$$= -2x + \frac{4}{3}x^3 + O(x^5).$$

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a) Eftersom vi har x^4 i nämnaren, skriver vi Maclaurinutvecklingen av täljaren t o m ordning 4:

$$\cos(x^2) - 1 = \left[\cos x = 1 - \frac{x^2}{2} + O(x^4)\right] =$$

$$= \cancel{1} - \frac{x^4}{2} + O(x^8) - \cancel{1} = -\frac{x^4}{2} + O(x^8).$$

Observera att vi kan skriva

$$O(x^8) \overset{\text{def av } O(x^8)}{=} x^8 \cdot \underbrace{f(x)}_{\text{begränsad då } x \text{ är liten}} = x^4 \cdot \overbrace{\underbrace{(x^4 f(x))}_{\text{begränsad då } x \text{ är liten}}}^{=O(x^4)} =$$

$$\underset{\text{det av } O(x^4)}{=} x^4 \cdot O(x^4).$$

Slutligen, $\displaystyle\lim_{x\to 0} \frac{\cos(x^2)-1}{x^4} = \lim_{x\to 0} \frac{-\frac{x^4}{2} + x^4\cdot O(x^4)}{x^4} =$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left(-\frac{1}{2}+O(x^4)\right)}{\cancel{x^4}} =$$

$$= \lim_{x\to 0}\left(-\frac{1}{2}+O(x^4)\right) = \lim_{x\to 0}\left(-\frac{1}{2}+\overbrace{x^4}^{\to 0}\cdot \underbrace{g(x)}_{\substack{\text{begränsad}\\ \text{då } x \text{ är liten}}}\right) =$$

$$= -\frac{1}{2}+0 = -\frac{1}{2}.$$

b) Låt oss först ta reda på den ledande termen i nämnaren:

$$\ln(1-x^2) = \left[\begin{array}{l}\ln(1+x) = \\ = x - \frac{x^2}{2} + O(x^3)\end{array}\right] = -x^2 + O(x^4).$$

Nu utvecklar vi täljaren t.o.m. ordning 2:

$$x\cdot\sin 2x = \left[\begin{array}{l}\sin x = \\ = x - \frac{x^3}{3!} + O(x^5)\end{array}\right] = x\cdot\left(2x + O(x^3)\right) =$$

$$= 2x^2 + O(x^4) \Longrightarrow$$

$$\lim_{x\to 0}\frac{x\sin 2x}{\ln(1-x^2)} = \lim_{x\to 0}\frac{2x^2+O(x^4)}{-x^2+O(x^4)} = \left[\begin{array}{c}\text{se del} \\ \text{a)}\end{array}\right] =$$

$$= \lim_{x\to 0}\frac{2x^2+x^2\cdot O(x^2)}{-x^2+x^2\cdot O(x^2)} = \lim_{x\to 0}\frac{\cancel{x^2}\left(2+O(x^2)\right)}{\cancel{x^2}\left(-1+O(x^2)\right)} =$$

$$= \lim_{x\to 0}\frac{2+\overbrace{x^2}^{0}\cdot\overbrace{f(x)}^{\text{begränsad, då } x\to 0}}{-1+\underbrace{x^2}_{0}\cdot\underbrace{g(x)}_{\substack{\text{begränsad,}\\ \text{då } x\to 0}}} = \frac{2+0}{-1+0} = -2.$$

c) Täljaren: $e^x - 1 - x = \cancel{1} + \cancel{x} + \frac{x^2}{2!} + O(x^3) - \cancel{1} - \cancel{x}$

$$= \frac{x^2}{2} + O(x^3)$$

Nämnaren: $(\arctan x)^2 = (x + O(x^3))^2 =$

$$= x^2 + 2x\,O(x^3) + (O(x^3))^2 =$$

$$= x^2 + O(x^4) + O(x^6)$$

$$= x^2 + O(x^4)$$

$$\Rightarrow \lim_{x \to 0} \frac{e^x - 1 - x}{(\arctan x)^2} = \lim_{x \to 0} \frac{\frac{x^2}{2} + O(x^3)}{x^2 + O(x^4)} =$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2} + x^2 \cdot O(x)}{x^2 + x^2 \cdot O(x^2)} =$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2} + O(x)\right)}{\cancel{x^2}\left(1 + O(x^2)\right)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} + \overbrace{x}^{\to 0} \cdot \overbrace{f(x)}^{\text{begräns. då } x \to 0}}{1 + \underbrace{x^2}_{\to 0} \cdot \underbrace{g(x)}_{\text{begr. då } x \to 0}} = \frac{\frac{1}{2}}{1} = \frac{1}{2},$$

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a) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} =$

$$= \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3!} + O(x^5)\right)}{x\left(x - \frac{x^3}{3!} + O(x^5)\right)} =$$

$$= \lim_{x \to 0} \frac{\cancel{x} - \cancel{x} + \frac{x^3}{6} + O(x^5)}{x^2 + O(x^4)} =$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{6} + x^3 \cdot O(x^2)}{x^2 + x^2 O(x^2)} = \lim_{x\to 0} \frac{x^3\left(\frac{1}{6} + O(x^2)\right)}{x^2\left(1 + O(x^2)\right)} =$$

$$= \lim_{x\to 0} \underbrace{x}_{\to 0} \cdot \frac{\frac{1}{6} + \overbrace{x^2}^{\to 0} \cdot \overbrace{f(x)}^{\text{begr.}}}{1 + \underbrace{x^2}_{\to 0} \cdot \underbrace{g(x)}_{\text{begr.}}} = 0 \cdot \frac{1}{6} = 0$$

b) $$\lim_{x\to 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right) = \lim_{x\to 0} \frac{x - \ln(1+x)}{x \ln(1+x)} =$$

$$= \lim_{x\to 0} \frac{x - \left(x - \frac{x^2}{2} + O(x^3)\right)}{x\left(x - \frac{x^2}{2} + O(x^3)\right)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \cancel{x} + \frac{x^2}{2} + O(x^3)}{x^2 + O(x^3)} = \lim_{x\to 0} \frac{\frac{x^2}{2} + x^3 \cdot \overbrace{f(x)}^{\text{begr.}}}{x^2 + x^3 \cdot \underbrace{g(x)}_{\text{begr.}}} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}\left(\frac{1}{2} + \overbrace{x}^{\to 0} \cdot \overbrace{f(x)}^{\text{begr.}}\right)}{\cancel{x^2}\left(1 + \underbrace{x}_{\to 0} \cdot \underbrace{g(x)}_{\text{begr.}}\right)} = \frac{1}{2}$$

c) $$\lim_{x\to\infty} \frac{\sqrt{x^2-1} - x\cos\frac{1}{x}}{\sin\frac{1}{x} - \tan\frac{1}{x}} =$$

OBS! Alla Maclaurin-utvecklingar gäller för x nära 0 $\Rightarrow$ sätter $y = \frac{1}{x} \to 0$

$$= \lim_{y\to 0} \frac{\sqrt{\frac{1}{y^2} - 1} - \frac{1}{y}\cos y}{\sin y - \tan y} =$$

$$= \lim_{y\to 0} \frac{\frac{1}{y}\sqrt{1-y^2} - \frac{1}{y}\cos y}{\sin y\left(1 - \frac{1}{\cos y}\right)} =$$

$$= \lim_{y\to 0} \frac{\frac{1}{y}\left(\sqrt{1-y^2} - \cos y\right) \cdot \cos y}{\sin y\,(\cos y - 1)} =$$

$$= \left[\begin{array}{l} (1+y)^{\frac{1}{2}} = 1 + \frac{1}{2}y + \binom{\frac{1}{2}}{2} y^2 + O(y^3) = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + O(y^3) \\ (1-y^2)^{\frac{1}{2}} = 1 - \frac{1}{2}y^2 - \frac{1}{8}y^4 + O(y^6) \end{array} \right]$$

$$= \lim_{y \to 0} \frac{\frac{1}{y}\left(\cancel{1} - \cancel{\frac{1}{2}y^2} - \frac{1}{8}y^4 + O(y^6) - \left[\cancel{1} - \cancel{\frac{y^2}{2}} + \frac{y^4}{4!} + O(y^6)\right]\right)}{\left(y - \frac{y^3}{3!} + O(y^5)\right)\left(\cancel{1} - \frac{y^2}{2} + \frac{y^4}{4!} + O(y^6) - \cancel{1}\right)}$$

$$= \lim_{y \to 0} \frac{\frac{1}{y}\left(-\frac{1}{8}y^4 - \frac{1}{24}y^4 + O(y^6)\right)}{-\frac{y^3}{2} + O(y^4)} =$$

$$\frac{1}{8} + \frac{1}{24} = \frac{3+1}{24} = \frac{1}{6}$$

$$= \lim_{y \to 0} \frac{-\frac{1}{6}y^4 + O(y^6)}{-\frac{y^4}{2} + O(y^5)} =$$

$$= \lim_{y \to 0} \frac{-\frac{1}{6}y^4 + y^6 \cdot \overbrace{f(x)}^{\text{begr.}}}{-\frac{y^4}{2} + y^5 \cdot \underbrace{g(x)}_{\text{begr.}}} = \lim_{y \to 0} \frac{\cancel{y^4}\left(-\frac{1}{6} + \overbrace{y^2}^{\to 0} \overbrace{f(x)}^{\text{begr.}}\right)}{\cancel{y^4}\left(-\frac{1}{2} + \underbrace{y}_{\to 0} \cdot \underbrace{g(x)}_{\text{begr.}}\right)} =$$

$$= \frac{-\frac{1}{6}}{-\frac{1}{2}} = \frac{1}{6} \cdot \frac{2}{1} = \underline{\frac{1}{3}}$$

Extra

$\boxed{B8}$ $\underline{10}$

$$\lim_{x \to 0^+} \frac{e^{ax} - \cos\sqrt{x}}{(\arctan x)^2} = \left[\begin{array}{l} \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + O(x^6) \Rightarrow \\ \cos\sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{24} + O(x^3) \end{array} \right]$$

$$= \lim_{x \to 0^+} \frac{\cancel{1} + ax + \frac{a^2x^2}{2} + O(x^3) - \cancel{1} + \frac{x}{2} - \frac{x^2}{24} + O(x^3)}{(x + O(x^3))(x + O(x^3))} =$$

$$= \lim_{x \to 0^+} \frac{x\left(a + \frac{1}{2}\right) + \left(\frac{a^2}{2} - \frac{1}{24}\right)x^2 + O(x^3)}{x^2 + O(x^4)} = \otimes$$

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Om $a + \frac{1}{2} \neq 0 \Rightarrow$

$$\otimes = \lim_{x \to 0^+} \frac{\cancel{x}\left(\left(a+\frac{1}{2}\right) + \left(\frac{a^2}{2} - \frac{1}{24}\right)x + O(x^2)\right)}{x^{\cancel{2}}\left(1 + \underbrace{O(x^2)}_{\to 0}\right)} =$$

$$= \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{a + \frac{1}{2} + \overbrace{O(x)}^{\to 0}}{1 + \underbrace{O(x^2)}_{\to 0}} = \frac{a+\frac{1}{2}}{1} \lim_{x \to 0^+} \frac{1}{x} = \infty$$

Det betyder att $a = -\frac{1}{2}$ annars blir inte gränsvärdet ändligt.

Om $a = -\frac{1}{2} \Rightarrow$

$$\otimes = \lim_{x \to 0^+} \frac{\left(\frac{1}{8} - \frac{1}{24}\right)x^2 + O(x^3)}{x^2 + O(x^4)} =$$

$$= \lim_{x \to 0^+} \frac{\cancel{x^2}\left[\left(\frac{1}{8} - \frac{1}{24}\right) + \overbrace{O(x)}^{\to 0}\right]}{\cancel{x^2}\left[1 + \underbrace{O(x^2)}_{\to 0}\right]} = \frac{1}{8} - \frac{1}{24} =$$

$$= \frac{3-1}{24} = \frac{2}{24} = \frac{1}{12}.$$