Lektion 7: definition av derivata

[P4]

1 Derivatan är definierad som

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2 a) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)' = \left(x^{1/2} + x^{-1/2}\right)' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-\frac{3}{2}} =$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}.$$

b) $\left((x - x^3)^{11}\right)' = 11(x - x^3)^{10}(1 - 3x^2)$

c) $\left(\frac{1+x^2}{1-x^2}\right)' = \frac{2x(1-x^2) - (-2x)(1+x^2)}{(1-x^2)^2} =$

$$= \frac{2x - \cancel{2x^3} + 2x + \cancel{2x^3}}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}$$

d) $(\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

e) $(\ln|4x|)' = \frac{1}{4x} \cdot 4 = \frac{1}{x}$

f) $(x^2 \ln x)' = 2x \ln x + \frac{x^2}{x} = 2x \ln x + x$

g) $\left(e^{-\frac{2}{x}}\right)' = e^{-\frac{2}{x}} \cdot \frac{2}{x^2}$

h) $(\ln(1+x^2))' = \frac{2x}{1+x^2}$

3 a) $\left(\ln \frac{1+x}{1-x}\right)' = \frac{1-x}{1+x} \cdot \frac{1-\cancel{x}+1+\cancel{x}}{(1-x)^2} = \frac{\cancel{1-x}}{1+x} \cdot \frac{2}{(\cancel{1-x})(1-x)}$

$$= \frac{2}{1-x^2}$$

b) $\left((\ln x)^3\right)' = 3(\ln x)^2 \cdot \frac{1}{x}$

c) $\left(\ln \frac{x}{\sqrt{1+x^2}}\right)' = \frac{\sqrt{1+x^2}}{x} \cdot \left(x \cdot (1+x^2)^{-1/2}\right)' =$

$= \frac{\sqrt{1+x^2}}{x} \cdot \left((1+x^2)^{-1/2} - \cancel{\frac{1}{2}} (1+x^2)^{-3/2} \cdot \cancel{2} x^2 \right)$

$= \frac{1}{x} - \frac{x}{1+x^2} = \frac{1+x^2-x^2}{x(1+x^2)} = \frac{1}{x(1+x^2)}$

d) $\left(e^{\sqrt{1+\ln x}}\right)' = e^{\sqrt{1+\ln x}} \cdot \frac{\frac{1}{x}}{2\sqrt{1+\ln x}} = \frac{e^{\sqrt{1+\ln x}}}{2x\sqrt{1+\ln x}}$

e) $\left(x e^{-\frac{1}{\sqrt{x}}}\right)' = e^{-\frac{1}{\sqrt{x}}} + x \cdot e^{-\frac{1}{\sqrt{x}}} \left(-x^{-1/2}\right)' =$

$= e^{-\frac{1}{\sqrt{x}}} + x \cdot e^{-\frac{1}{\sqrt{x}}} \cdot \frac{1}{2} x^{-3/2} =$

$= e^{-\frac{1}{\sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)$

g) $\left(e^{-x} \cos x\right)' = -e^{-x} \cos x - e^{-x} \sin x =$

$= -e^{-x} (\sin x + \cos x)$

f) $\left(\ln(x + \sqrt{1+x^2})\right)' = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right) =$

$= \frac{1}{\sqrt{1+x^2}}$

h) $(\tan x - x)' = \frac{1}{\cos^2 x} - 1 = \left(\frac{\sin x}{\cos x}\right)^2 = (\tan x)^2$

i) $\left(\ln\left|\tan \frac{x}{2}\right|\right)' = \frac{1}{\tan \frac{x}{2}} \cdot \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2} =$

$= \frac{\cancel{\cos \frac{x}{2}}}{\sin \frac{x}{2}} \cdot \frac{1}{\cos^{\cancel{2}} \frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{\sin x}$

j) $\left((\arctan x)^2\right)' = 2\arctan x \cdot \dfrac{1}{1+x^2}$

k) $\left(\arctan \dfrac{4}{x}\right)' = \dfrac{1}{1+\left(\frac{4}{x}\right)^2} \cdot \left(-\dfrac{4}{x^2}\right) =$

$$= -\frac{\frac{4}{x^2}}{1+\frac{16}{x^2}} = -\frac{4}{x^2+16}$$

l) $\left(\arcsin(x^2-1)\right)' = \dfrac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x =$

$$= \frac{2x}{\sqrt{\cancel{1}-x^4+2x^2-\cancel{1}}} = \frac{2x}{\sqrt{2x^2-x^4}} .$$

4 a) kontinuerlig men inte deriverbar i $x=2$:

$f(x)=|x-2|$: $\lim\limits_{x\to 2} |x-2| = 0$, men

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$ existerar inte,

eftersom

$$\lim_{h\to 0+} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \text{ , medan}$$

$$\lim_{h\to 0-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1 .$$

b) Alla deriverbara i en punkt funktioner är också kontinuerliga i denna punkt.
T ex antag att $\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$ existerar och ändlig, då

$\lim\limits_{h\to 0} \left(f(2+h) - f(2)\right) = \lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} \cdot h =$ 3

$$= \lim_{h\to 0} f'(2)\cdot \underbrace{h}_{\to 0} = 0 \Rightarrow \lim_{h\to 0} f(2+h) = f(2)$$

6

En linje som går genom punkten $(1, 2)$ har ekvationen $y = 2 + k(x-1)$ där k är linjens lutning.

För tangenten ges lutningen av derivatan:

om $y = x + \sqrt{x} \Rightarrow k = (x + \sqrt{x})'\big|_{x=1} =$

$$= 1 + \frac{1}{2\sqrt{x}}\bigg|_{x=1} = 1 + \frac{1}{2}$$

$$= \frac{3}{2} \Rightarrow$$

tangentens ekvation är $y = 2 + \frac{3}{2}(x-1)$.

Normalens lutning är $-\frac{1}{f'(2)} =$

$= -\frac{2}{3}$, så

normalens ekvation är $y = 2 - \frac{2}{3}\cdot(x-1)$.

Detta kan förenklas:

tangenten: $y = \frac{3}{2}x + \frac{1}{2}$
normalen: $y = -\frac{2}{3}x + \frac{8}{3}$

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$\underline{\underline{2^{b}}}$ $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=$$

$$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\cancel{x}-\cancel{x}-h}{(x+h)x\cdot h}=$$

$$=\lim_{h\to 0}\frac{-\cancel{h}}{x(x+h)\cancel{h}}=-\frac{1}{x^2}.$$

$\underline{\underline{6^{a}}}$ $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=$$

$$=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x+h})\sqrt{x}}=$$

$$=\lim_{h\to 0}\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=$$

$$=\lim_{h\to 0}\frac{\cancel{x}-\cancel{x}-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=$$

$$=\lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=-\frac{1}{2(\sqrt{x})^3}=$$

$$=-\frac{1}{2x\sqrt{x}}.$$

Extra

5 Om $y = f(x)$ så differentialen är
$dy = f'(x)\,dx$.

a) $y = x^3 \Rightarrow dy = (x^3)'dx$, där

$$(x^3)' = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = 3x^2$$

Så $dy = 3x^2dx$

b) $y = \dfrac{x}{x+1} \Rightarrow dy = \left(\dfrac{x}{x+1}\right)'dx$ där

$$\left(\frac{x}{x+1}\right)' = \lim_{h\to 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} =$$

$$= \lim_{h\to 0} \frac{(x+h)(x+1) - x(x+h+1)}{h(x+1)(x+h+1)} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{xh} + \cancel{x} + h - \cancel{x^2} - \cancel{xh} - \cancel{x}}{h(x+1)(x+h+1)} = \frac{1}{(x+1)^2}$$

Så $dy = \dfrac{dx}{(x+1)^2}$.