Lektion 19

$\boxed{P6}$ **1** a) Beräknar först en primitiv funktion!

$$\int e^{2x}dx = \frac{1}{2}e^{2x} \Rightarrow$$

$$\int_0^1 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}e^2 - \frac{1}{2}$$

c) $$\int_0^1 \frac{2x}{x^2+1}dx = \left[\ln(x^2+1)\right]_0^1 = \ln 2 - \ln 1 = \ln 2$$

h) $$\int_0^1 \frac{dx}{\sqrt{x^2+4}} = \left[\ln\left(x+\sqrt{x^2+4}\right)\right]_0^1 = \ln(1+\sqrt{5}) - \ln 2 =$$

$$= \ln\frac{1+\sqrt{5}}{2}$$

k) Beräknar först $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx =$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + C \Rightarrow$$

$$\int_0^{\pi} \sin^2 x\,dx = \left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{\pi}{2} - \overbrace{\frac{\sin 2\pi}{4}}^{=0} - 0 = \frac{\pi}{2}$$

7 d) $$\int(9x^2+4x)\ln x\,dx =$$

$$= (3x^3+2x^2)\ln x - \int(3x^2+2x)dx =$$

$$= (3x^3+2x^2)\ln x - x^3 - x^2 \Rightarrow$$

$$\int_1^3 (9x^2+4x)\ln x\,dx = \left[(3x^3+2x^2)\ln x - x^3 - x^2\right]_1^3 =$$

$$= (81+18)\ln 3 - 27 - 9 - (3+2)\cdot 0 + 1 + 1 =$$

$$= 99\ln 3 - 34$$

h) $\int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \left[\begin{matrix} \sqrt{x}=t, & x=t^2 \\ & dx = 2t\,dt \end{matrix} \right] =$

$= \int \frac{2\cancel{t}\,dt}{\cancel{t}(t+1)} = \int \frac{2dt}{t+1} = 2\ln(t+1) = 2\ln(\sqrt{x}+1) \Rightarrow$

$\int_1^2 \frac{dx}{x+\sqrt{x}} = \left[2\ln(\sqrt{x}+1)\right]_1^2 = 2\ln(1+\sqrt{2}) - 2\ln 2 =$

$= 2\ln\frac{\sqrt{2}+1}{2}$

j) $\int \cos\sqrt[3]{x}\,dx = \left[\begin{matrix} \sqrt[3]{x} = t \\ x = t^3 \Rightarrow dx = 3t^2dt \end{matrix} \right] =$

$= \int \cos t \cdot 3t^2 dt = 3\int t^2 \cos t\, dt =$

$= 3\left[t^2 \sin t - 2\int t \sin t\, dt \right] =$

$= 3\left[t^2 \sin t - 2\left(-t\cos t - \int(-\cos t)dt\right)\right] =$

$= 3\left[t^2 \sin t + 2t\cos t - 2\int \cos t\, dt\right] =$

$= 3(t^2\sin t + 2t\cos t - 2\sin t) + C$

$= 3\left(\sqrt[3]{x^2}\sin\sqrt[3]{x} + 2\sqrt[3]{x}\cos\sqrt[3]{x} - 2\sin\sqrt[3]{x}\right) + C$

$\int_0^1 \cos\sqrt[3]{x}\,dx = 3(1\cdot\sin 1 + 2\cos 1 - 2\sin 1 - 0 - 0 - 0)$

$= -3\sin 1 + 6\cos 1$

k) $\int \cos^5 x\, dx = \int \cos^4 x \cdot \cos x\, dx = \int (1-\sin^2 x)^2 \cos x\, dx =$

$= \left[\begin{matrix} \sin x = t \\ dt = \cos x\, dx \end{matrix} \right] = \int (1-t^2)^2 dt =$

$= \int (1 - 2t^2 + t^4)\, dt = t - \frac{2t^3}{3} + \frac{t^5}{5} + C =$

$= \sin x - \frac{2(\sin x)^3}{3} + \frac{(\sin x)^5}{5} + C \Rightarrow$

$$\int_0^{\pi} \cos^5 x\,dx = \left[\sin x - \frac{2(\sin x)^3}{3} + \frac{(\sin x)^5}{5}\right]_{x=0}^{x=\pi} = \underline{\underline{0}}$$

$\sin\pi = 0$
$\sin 0 = 0$

6 a)

$$\underset{0 \le x \le 2\pi}{|\sin x|} = \begin{cases} \sin x & 0 \le x \le \pi \\ -\sin x & -\pi \le x \le 2\pi \end{cases}$$

$$\int_0^{2\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx =$$

$$= \left[-\cos x\right]_0^{\pi} + \left[\cos x\right]_{\pi}^{2\pi} = 1 + 1 + 1 + 1 = \underline{4}$$

b)

$$\underset{0 \le x \le 2}{|x^3 - 1|} = \begin{cases} -(x^3 - 1) & 0 \le x \le 1 \\ x^3 - 1 & 1 < x \le 2 \end{cases} \Rightarrow$$

$$\int_0^2 |x^3 - 1|\,dx = \int_0^1 (1 - x^3)\,dx + \int_1^2 (x^3 - 1)\,dx =$$

$$= \left[x - \frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - x\right]_1^2 =$$

$$= \cancel{1} - \frac{1}{4} + 4 - \cancel{2} - \frac{1}{4} + \cancel{1} = 3\frac{1}{2} = \underline{\frac{7}{2}}$$

9 a)

$$\int_2^4 \frac{x^2}{x - 1}\,dx = \int_2^4 \left(x + 1 + \frac{1}{x-1}\right)dx =$$

$$x^2 : (x - 1) = x + 1$$
$$-(x^2 - x)$$
$$x$$
$$-(x - 1)$$
$$1$$

$$= \left[\frac{x^2}{2} + x + \ln(x - 1)\right]_2^4 =$$

$$= 8 + \cancel{4} + \ln 3 - \cancel{2} - \cancel{2} - \cancel{\ln 1} =$$

$$= \underline{8 + \ln 3}$$

c) Beräknar först

$$\int \frac{dx}{x(x^2 + 3x + 2)} = \int \frac{dx}{x(x+1)(x+2)} = \otimes$$

Partialbråkuppdelning:

$$\int \frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$, handpåläggning ger

$$A = \frac{1}{2}, \quad B = -1, \quad C = \frac{1}{2} \Rightarrow$$

$$\otimes = \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C$$

$$\Rightarrow \int_1^2 \frac{dx}{x(x^2+3x+2)} = \frac{1}{2}\ln 2 - \ln 3 + \overbrace{\frac{1}{2}\ln \underbrace{4}_{2^2}}^{=\ln 2} - 0 + \ln 2 - \frac{1}{2}\ln 3 =$$

$$= \frac{5}{2}\ln 2 - \frac{3}{2}\ln 3$$

f) Partialbråksuppdelar

$$\frac{x-1}{(x+1)^2(x^2+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+3} \qquad (*)$$

Handpåläggning ger att $B = \frac{-2}{4} = -\frac{1}{2}$.

Vidare,

$Cx^2 + Dx + Cx + D$

$$(**) \quad \underbrace{\frac{x-1}{(x+1)(x^2+3)}}_{\to 0} = A + \underbrace{\frac{B}{x+1}}_{\to 0} + \underbrace{\frac{(Cx+D)(x+1)}{x^2+3}}_{\to C} \quad \text{då } x \to \infty$$

$$\Rightarrow \boxed{A + C = 0}$$

Låt $x = 0$ i $(*)$ $\Rightarrow -\frac{1}{3} = A + \overbrace{B}^{=-\frac{1}{2}} + \frac{D}{3} \Rightarrow \boxed{3A + D = \frac{1}{2}}$

Låt $x = 1$ i $(**)$ $\Rightarrow 0 = A + \frac{\overbrace{B}^{=-\frac{1}{2}}}{2} + \frac{C+D}{2}$

$$\Rightarrow \boxed{\frac{1}{2} = 2A + C + D}$$

Löser systemet:

$$\begin{cases} C = -A \\ D = \frac{1}{2} - 3A \\ \frac{1}{2} = 2A + \underbrace{C}_{-A} + \underbrace{D}_{\frac{1}{2} - 3A} \end{cases} \Rightarrow \begin{cases} C = 0 \\ D = \frac{1}{2} \\ A = 0 \end{cases} \text{ och } B = -\frac{1}{2} \Rightarrow$$

$$\int \frac{x-1}{(x+1)^2(x^2+3)}\,dx = \int\left(-\frac{1}{2(x+1)^2} + \frac{1}{2(x^2+3)}\right)dx$$

$$= \frac{1}{2}(x+1)^{-1} + \frac{1}{6}\int \frac{dx}{\left(\frac{x}{\sqrt{3}}\right)^2+1} = \frac{1}{2}(x+1)^{-1} + \frac{\sqrt{3}}{6}\arctan\frac{x}{\sqrt{3}} + C$$

$$\Rightarrow \int_0^1 \frac{x-1}{(x+1)^2(x^2+3)}\,dx = \frac{1}{4} + \frac{\sqrt{3}}{6}\arctan\frac{1}{\sqrt{3}} - \frac{1}{2} =$$

$$= -\frac{1}{4} + \frac{\sqrt{3}}{6}\cdot\frac{\pi}{6} = -\frac{1}{4} + \frac{\pi\sqrt{3}}{36}.$$

Extra $\boxed{P6}$

9g $\quad I = \int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)}$, dar

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$1 = Ax^2 - Ax + A + Bx^2 + Cx + Bx + C$$

$$1 = (A+B)x^2 + (-A+C+B)x + A + C \Rightarrow$$

$$\begin{cases} A+B=0 \\ -A+C+B=0 \\ A+C=1 \end{cases} \qquad \begin{aligned} B &= -A \\ C &= 1-A \\ -A+1-A-A &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} B &= -\tfrac{1}{3} \\ C &= \tfrac{2}{3} \\ A &= \tfrac{1}{3} \end{aligned}$$

$$\Rightarrow I = \frac{1}{3}\ln|x+1| + \underbrace{\frac{1}{3}\int \frac{-x+2}{x^2-x+1}\,dx}_{=I_1}$$

$$I_1 = -\frac{1}{3}\int \frac{x-2}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx = -\frac{1}{3}\cdot\frac{4}{3}\int \frac{x-2}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\,dx =$$

$$= \left[\begin{aligned} \frac{2x-1}{\sqrt{3}} &= t \\ dt &= \frac{2}{\sqrt{3}}\,dx \\ x &= \frac{\sqrt{3}}{2}t + \frac{1}{2} \end{aligned}\right] = -\frac{\cancel{4}}{9}\int \frac{\frac{\sqrt{3}t+1-4}{\cancel{2}}}{t^2+1}\,\frac{\sqrt{3}}{\cancel{2}}\,dt =$$

$$= -\frac{1}{3}\int \frac{t}{t^2+1}\,dt + \frac{\sqrt{3}}{3}\int \frac{dt}{t^2+1} =$$

$$= -\frac{1}{6}\ln\left(t^2+1\right) + \frac{\sqrt{3}}{3}\arctan t + C =$$

$$= -\frac{1}{6}\ln\left(\frac{4x^2-4x+4}{3}\right) + \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C =$$

$$= -\frac{1}{6}\ln\left(4x^2-4x+4\right) + \underbrace{\frac{1}{6}\ln 3}_{=\text{konst}} + \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

$$\Rightarrow \int_0^1 \frac{dx}{x^3+1} = \left[\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(4x^2-4x+4\right) + \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}}\right]_0^1 =$$

$$= \frac{1}{3}\ln 2 - \cancel{\frac{1}{6}\ln 4} + \cancel{\frac{1}{6}\ln 4} + \frac{\sqrt{3}}{3}\arctan\frac{1}{\sqrt{3}} +$$

$$+ \frac{\sqrt{3}}{3}\arctan\left(+\frac{1}{\sqrt{3}}\right) =$$

$$= \frac{\ln 2}{3} + \frac{\cancel{2}\sqrt{3}}{3}\cdot\frac{\pi}{\cancel{6}\,3} = \frac{\ln 2}{3} + \frac{\pi\sqrt{3}}{9}.$$

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$$\int \frac{x\ln x}{(1+x^2)^2}\,dx = \left[\int \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}(1+x^2)^{-1}\right]$$

$$= -\frac{1}{2}(1+x^2)^{-1}\ln x + \frac{1}{2}\int \frac{1}{x(1+x^2)}\,dx =$$

$$= \left[\begin{array}{l} \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \Rightarrow 1 = A + Ax^2 + Bx^2 + Cx \\ \qquad 1 = (A+B)x^2 + Cx + A \\ \qquad \Rightarrow A = 1,\ C = 0,\ B = -1 \end{array}\right] =$$

$$= -\frac{1}{2}(1+x^2)^{-1}\ln x + \frac{1}{2}\int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx =$$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\ln x - \frac{1}{4}\ln(1+x^2) + C \Rightarrow$$

$$\int_{1/2}^{2} \frac{x\ln x}{(1+x^2)^2}\,dx = -\frac{\ln 2}{10} + \frac{1}{2}\ln 2 - \frac{1}{4}\ln 5 + \frac{\ln\frac{1}{2}}{2(1+\frac{1}{4})} - \frac{1}{2}\ln\frac{1}{2}$$

$$+ \frac{1}{4}\ln\frac{5}{4} = -\frac{\ln 2}{10} + \cancel{\frac{\ln 2}{2}} - \cancel{\frac{\ln 5}{4}} - \frac{4}{10}\ln 2 + \cancel{\frac{\ln 2}{2}}$$

$$+ \cancel{\frac{\ln 5}{4}} - \frac{\cancel{2}\ln 2}{\cancel{4}\,2} = 0.$$