Lektion 18

$\boxed{P5}$ **26** Beräkna $\int \sqrt{1-x^2}\,dx$

a) genom variabelbytet $x = \sin t \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$!

$$\int \sqrt{1-x^2}\,dx = \begin{bmatrix} x = \sin t & dt = \dfrac{dx}{\sqrt{1-x^2}} \\ t = \arcsin x & \end{bmatrix} =$$

$$= \int (1-x^2)\underbrace{\frac{dx}{\sqrt{1-x^2}}}_{=dt} = \int (1-\sin^2 t)\,dt =$$

$$= \int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{1}{2}t + \frac{\sin 2t}{4} + C = \frac{1}{2}\arcsin x + \frac{\sin(2\arcsin x)}{4} + C$$

$$= \frac{1}{2}\arcsin x + \frac{2\sin(\arcsin x)\cos(\arcsin x)}{\cancel{4}\,2} + C =$$

$$= \begin{bmatrix} \text{OBS! } \arcsin x \in [-\frac{\pi}{2}; \frac{\pi}{2}] \Rightarrow \cos(\arcsin x) \ge 0 \\ \Rightarrow \cos(\arcsin x) = \sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2} \end{bmatrix} =$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

b) Partiel integration!

skriver om som $a\cdot(1-x^2)+b$

$$\int \sqrt{1-x^2}\cdot 1\,dx = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{dx}{\sqrt{1-x^2}} =$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \arcsin x + C \Rightarrow$$

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \arcsin x + C \Rightarrow$$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

27 a) $$\int \frac{\sqrt{x-2}}{x-1}\,dx = \left[\begin{array}{l} \sqrt{x-2} = t \Rightarrow x = t^2+2 \\ \qquad\qquad\quad dx = 2t\,dt \end{array} \right] =$$

$$= \int \frac{t}{t^2+1}\,2t\,dt = 2\int \frac{t^2}{t^2+1}\,dt =$$

$$\begin{array}{r|l} 1 & \\ t^2 & t^2+1 \\ -\,\underline{t^2+1} & \\ -1 & \end{array}$$

$$= 2\int \left(1 - \frac{1}{t^2+1}\right)dt = 2t - 2\arctan t + C =$$

$$= 2\sqrt{x-2} - 2\arctan\sqrt{x-2} + C$$

b) $$\int \frac{\sqrt[3]{x}}{x-1}\,dx = \left[\begin{array}{l} \sqrt[3]{x} = t \Rightarrow x = t^3 \\ \qquad\qquad\quad dx = 3t^2dt \end{array} \right] =$$

$$= \int \frac{t}{t^3-1}\,3t^2dt = 3\int \frac{t^3}{t^3-1}\,dt =$$

$$= 3\int \frac{t^3-1+1}{t^3-1}\,dt = 3\int\left(1 + \frac{1}{t^3-1}\right)dt =$$

$$= 3t + \underbrace{3\int \frac{dt}{t^3-1}}_{=I} = \otimes$$

Partialbråkuppdelning: $t^3-1 = (t-1)(t^2+t+1)$

$$\frac{1}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \quad (*)$$

Handpåläggning: $A = \frac{1}{3}$ $\left(\begin{array}{l} \cdot(t-1) \text{ hela,} \\ \text{och } t \to 1 \end{array}\right)$

Låt $t=0$ i $(*)$: $-1 = \frac{A}{-1} + \frac{C}{1} \Rightarrow$

$$-1 = -\frac{1}{3} + C \Rightarrow C = -\frac{2}{3}$$

$$t \cdot (*) \Rightarrow \frac{t}{(t-1)(t^2+t+1)} = \frac{t}{3(t-1)} + \frac{Bt^2 - \frac{2}{3}}{t^2+t+1}$$

$t \to \infty$: ... $\to 0$ ... $\frac{1}{3} \leftarrow \frac{1}{3(\ldots)}$ ... $= \frac{t^2(B - 2/3t^2)}{\ldots} \to B$

så att $\quad 0 = \frac{1}{3} + B \quad$ i gränsvärdet $\Rightarrow B = -\frac{1}{3}$.

$$I = 3\int\left(\frac{1}{3(t-1)} + \frac{-\frac{1}{3}t - \frac{2}{3}}{t^2+t+1}\right)dt =$$

$$= \int \frac{dt}{t-1} - \underbrace{\int \frac{t+1}{t^2+t+1}dt}_{I_1} = \ln|t-1| - I_1,$$

där $I_1 = \underbrace{\int \frac{t+2}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dt}_{\text{(kvadratkompl.)}} = \frac{4}{3}\int \frac{t+2}{\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}dt =$

$$= \frac{4}{3}\int \frac{t+2}{\left(\frac{2t+1}{\sqrt{3}}\right)^2+1}dt = \left[\begin{array}{r} \frac{2t+1}{\sqrt{3}} = y \Rightarrow t = \frac{\sqrt{3}y-1}{2} \\ dt = \frac{\sqrt{3}}{2}dy \end{array}\right]$$

$$= \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}y \overbrace{- \frac{1}{2} + 2}^{=\frac{3}{2}}}{y^2+1}\cdot\frac{\sqrt{3}}{2}dy =$$

$$= \frac{4}{3}\int \frac{\frac{3}{4}y}{y^2+1}dy + \frac{4}{3}\int \frac{\frac{3\sqrt{3}}{4}}{y^2+1}dy =$$

$$= \frac{1}{2}\ln(y^2+1) + \sqrt{3}\arctan y + C =$$

$$= \frac{1}{2}\ln\left(\frac{(2t+1)^2}{3}+1\right) + \sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{4t^2+4t+4}{3}\right) + \sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \quad I = \ln|t-1| - \frac{1}{2}\ln\left(\frac{4}{3}(t^2+t+1)\right) - \sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)$$

Slutligen, $t = \sqrt[3]{x} \Rightarrow$

$$\otimes = 3\sqrt[3]{x} + \ln|\sqrt[3]{x}-1| \underbrace{- \frac{1}{2}\ln\frac{4}{3}}_{\text{konstant, kan "absorberas" i } C} - \frac{1}{2}\ln(x^{2/3}+x^{1/3}+1)$$
$$- \sqrt{3}\arctan\frac{2\sqrt[3]{x}+1}{\sqrt{3}} + C$$

Svar: $3\sqrt[3]{x} + \ln|\sqrt[3]{x}-1| - \frac{1}{2}\ln\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right) -$
$- \sqrt{3}\arctan\frac{2\sqrt[3]{x}+1}{\sqrt{3}} + C$

c) $\int \frac{dx}{\sqrt{2x-x^2}} = \left[\begin{array}{l} \text{satsar på} \\ \int \frac{dy}{\sqrt{1-y^2}} = \arcsin y + C \end{array} \right] =$

$= \int \frac{dx}{\sqrt{1-1+2x-x^2}} = \int \frac{dx}{\sqrt{1-(1-x)^2}} =$

$= \left[\begin{array}{l} 1-x = y \\ dy = -dx \end{array} \right] = -\int \frac{-dx}{\sqrt{1-(1-x)^2}} = -\int \frac{dy}{\sqrt{1-y^2}} =$

$= -\arcsin y + C = \overbrace{-\arcsin(1-x)}^{=\arcsin(x-1)} + C$

f) $\int \frac{dx}{\sqrt{x^2+2x+2}} = \left[\begin{array}{l} \text{gör igen kvad-} \\ \text{ratkompletering} \end{array} \right] =$

$= \int \frac{dx}{\sqrt{(x+1)^2+1}} = \left[\begin{array}{l} x+1 = t \\ dx = dt \end{array} \right] = \int \frac{dt}{\sqrt{t^2+1}} =$

$= \ln|t + \sqrt{t^2+1}| + C = \ln \left(\overbrace{\underset{x+1}{x+1} + \underset{>|x+1|}{\sqrt{x^2+2x+2}}}^{>0} \right) + C$

h) $\int \frac{x}{\sqrt{x^2+2x+2}} dx = \int \frac{x}{\sqrt{(x+1)^2+1}} dx = \left[\begin{array}{l} x+1 = t \\ x = t-1 \\ dx = dt \end{array} \right] =$

$= \int \frac{t-1}{\sqrt{t^2+1}} dt = \int \frac{t\,dt}{\sqrt{t^2+1}} - \int \frac{dt}{\sqrt{t^2+1}} =$

$= \left[\left(\sqrt{t^2+1}\right)' = \frac{2t}{2\sqrt{t^2+1}} = \frac{t}{\sqrt{t^2+1}} \right] =$

$= \sqrt{t^2+1} - \ln|t + \sqrt{t^2+1}| + C =$

$= \sqrt{x^2+2x+2} - \ln\left(x+1+\sqrt{x^2+2x+2}\right) + C.$

28 a) $\int x\sqrt{x^4+2x^2+3}\,dx = \int \sqrt{(x^2+1)^2+2}\; x\,dx$

$= \sqrt{2} \cdot \int \sqrt{\left(\frac{x^2+1}{\sqrt{2}}\right)^2 + 1}\; x\,dx =$

$$= \left[\frac{x^2+1}{\sqrt{2}} = t \Rightarrow \frac{2x}{\sqrt{2}}\,dx = dt,\ dt = \sqrt{2}\,x\,dx \right] =$$

$$= \int \sqrt{\left(\frac{x^2+1}{\sqrt{2}}\right)^2+1} \cdot \underbrace{\sqrt{2}\,x\,dx}_{dt} = \int \sqrt{t^2+1}\,dt =$$

$$= \left[\begin{matrix} \text{partiel-} \\ \text{integr.} \end{matrix} \right] = \int 1 \cdot \sqrt{t^2+1}\,dt =$$

$$= t\sqrt{t^2+1} - \int \frac{t^2}{\sqrt{t^2+1}}\,dt = t\sqrt{t^2+1} - \int \frac{t^2+1-1}{\sqrt{t^2+1}}\,dt =$$

$$= t\sqrt{t^2+1} - \int \sqrt{t^2+1}\,dt + \int \frac{dt}{\sqrt{t^2+1}} =$$

$$= t\sqrt{t^2+1} + \ln\left|t+\sqrt{t^2+1}\right| - \int \sqrt{t^2+1} + C$$

$$\Rightarrow 2\int \sqrt{t^2+1} = t\sqrt{t^2+1} + \ln\left|t+\sqrt{t^2+1}\right| + C$$

$$\Rightarrow \int \sqrt{t^2+1} = \frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right| + C.$$

$$\Rightarrow \int x\sqrt{x^4+2x^2+3}\,dx = \frac{1}{2}\,\frac{x^2+1}{\sqrt{2}} \cdot \sqrt{\left(\frac{x^2+1}{\sqrt{2}}\right)^2+1} +$$

$$+ \frac{1}{2}\ln\left(\frac{x^2+1}{\sqrt{2}} + \sqrt{\left(\frac{x^2+1}{\sqrt{2}}\right)^2+1}\right) + C =$$

$$= \frac{1}{2\sqrt{2}}(x^2+1)\sqrt{\frac{x^2+2x+3}{2}} + \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}\left(x^2+1+\sqrt{x^2+2x+3}\right)\right)$$

$$= \frac{1}{4}(x^2+1)\sqrt{x^2+2x+3} + \frac{1}{2}\ln\left(x^2+1+\sqrt{x^2+2x+3}\right) +$$

$$+ \underbrace{\frac{1}{2}\ln\frac{1}{\sqrt{2}} + C}_{=C}$$

__Svar:__ $\frac{1}{4}(x^2+1)\sqrt{x^2+2x+3} + \frac{1}{2}\ln\left(x^2+1+\sqrt{x^2+2x+3}\right) + C$

в) $\int \cos x \sqrt{\cos 2x}\,dx = \int \cos x \sqrt{1-2\sin^2 x}\,dx =$

$$= \left[\begin{matrix} \sin x = t \\ dt = \cos x\,dx \end{matrix}\right] = \int \sqrt{1-2t^2}\,dt =$$

$$= \int \underset{\uparrow}{1} \cdot \underset{\downarrow}{\sqrt{1-2t^2}}\,dt = t\sqrt{1-2t^2} - \int \frac{-2t^2}{\sqrt{1-2t^2}}\,dt =$$

$$= t\sqrt{1-2t^2} - \int \frac{(-2t^2+1)-1}{\sqrt{1-2t^2}}\,dt =$$

$$= t\sqrt{1-2t^2} - \int \sqrt{1-2t^2}\,dt + \int \frac{dt}{\sqrt{1-2t^2}}$$

$$\Rightarrow\ 2\int \sqrt{1-2t^2}\,dt = t\sqrt{1-2t^2} + \int \frac{dt}{\sqrt{1-(\sqrt{2}t)^2}}$$

$$\int \sqrt{1-2t^2}\,dt = \frac{1}{2}t\sqrt{1-2t^2} + \frac{1}{\sqrt{2}}\arcsin(\sqrt{2}t).$$

$$\Rightarrow \int \cos x \sqrt{\cos 2x}\,dx = \frac{1}{2}t\sqrt{1-2t^2} + \frac{1}{\sqrt{2}}\arcsin \sqrt{2}t =$$

$$= \frac{1}{2}\sin x\sqrt{1-2\sin^2 x} + \frac{1}{\sqrt{2}}\arcsin(\sqrt{2}\cdot\sin x) + C$$

$$= \frac{1}{2}\sin x\sqrt{\cos 2x} + \frac{1}{\sqrt{2}}\arcsin(\sqrt{2}\cdot\sin x) + C$$

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$$\begin{array}{l} t^2+t \ \big|\ t-1 \quad = t+2 \\ -(t^2-t) \\ 2t \\ -(2t-2) \\ 2 \end{array}$$

c) $\int \frac{1+\sqrt{x+1}}{1-\sqrt{x+1}}\,dx = \left[\begin{matrix} \sqrt{x+1} = t \\ dt = \frac{dx}{2\sqrt{x+1}} \end{matrix}\right] =$

$$= \int \frac{(1+\sqrt{x+1})\cdot 2\sqrt{x+1}}{1-\sqrt{x+1}} \cdot \underbrace{\frac{dx}{2\sqrt{x+1}}}_{=dt} =$$

$$= \int \frac{(1+t)\cdot 2t}{1-t}\,dt = -2\int \frac{t^2+t}{t-1}\,dt =$$

$$= -2\int \left(t+2+\frac{2}{t-1}\right)dt = -2\left(\frac{t^2}{2}+2t+2\ln|t-1|\right)+C$$

konst

$$= -(x+\cancel{1}) - 4\sqrt{x+1} - 4\ln|\sqrt{x+1}-1| + C$$

Extra $\boxed{P5}$

38 Beräknar först

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x^2}} = \int \frac{dx}{x^{1/2}(1+x^{1/6})} = \left[\begin{array}{l} x^{1/6}=t \\ dt = \frac{1}{6}x^{-5/6} \end{array}\right] =$$

$$= \int \frac{6x^{5/6}\cdot \overbrace{\tfrac{1}{6}x^{-5/6}dx}^{=dt}}{x^{1/2}(1+x^{1/6})} =$$

$$= \int \frac{6t^5\,dt}{t^3(1+t)} = \int \frac{6t^2\,dt}{1+t} =$$

$$\begin{array}{rl} & t-1 \\ t^2 & \big|\ t+1 \\ -\ t^2+t & \\ \hline -t & \\ -\ -t-1 & \\ \hline 1 & \end{array}$$

$$= 6\int\left(t-1+\frac{1}{t+1}\right)dt =$$

$$= 6\left(\frac{t^2}{2}-t+\ln|t+1|\right) =$$

$$= 6\left(\frac{\sqrt[3]{x}}{2}-\sqrt[6]{x}+\ln\left(\sqrt[6]{x}+1\right)\right)+C \Rightarrow$$

$$f(x) = 3\sqrt[3]{x}-6\sqrt[6]{x}+6\ln(\sqrt[6]{x}+1)+C$$

$$\lim_{x\to 0}\frac{f(x)}{\sqrt[6]{x}} = \lim_{x\to 0}\left(\underbrace{\frac{3\,\cancel{\sqrt[3]{x}}^{\,\sqrt[6]{x}}}{\cancel{\sqrt[6]{x}}}}_{\to 0} - 6 + \underbrace{\frac{6\ln(\overset{\to 0}{\sqrt[6]{x}}+1)}{\sqrt[6]{x}_{\to 0}}}_{\to 6 \text{ (standart-gränsvärde)}} + \frac{C}{\sqrt[6]{x}}\right)$$

$= \infty$, om $C \neq 0$.

Slutsatsen är att $C = 0$. I så fall

$$\lim_{x\to 0}\frac{f(x)}{\sqrt[6]{x}} = 0-6+6 = \underline{\underline{0}}.$$