Lektion 17

P5

21 a) $\int \frac{\sin x \cos x}{2-\sin^2 x}\,dx = \left[\begin{array}{l}\sin x = t \\ dt = \cos x\,dx\end{array}\right] =$

$$= \int \frac{t\,dt}{2-t^2} = -\frac{1}{2}\ln|2-t^2| + C =$$

$$= -\frac{1}{2}\ln(2-\sin^2 x) + C.$$

b) $\int \sin^4 x\,dx$ – se t ex boken s 263 för liknande problem

$$\boxed{\sin x = \frac{e^{ix}-e^{-ix}}{2i}} \Rightarrow$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$\sin^4 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4 =$$

$$= \frac{(e^{ix})^4 + 4(e^{ix})^3(-e^{-ix}) + 6(e^{ix})^2(e^{-ix})^2 + 4(e^{ix})(-e^{-ix})^3 + (-e^{-ix})^4}{(2i)^4}$$

$$= \frac{e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix}}{16} =$$

$$= \frac{e^{4ix}+e^{-4ix}}{16} - \frac{\cancel{4}\,1}{\cancel{16}\,4}\cdot(e^{2ix}+e^{-2ix}) + \frac{6}{16} =$$

$$= \boxed{\cos x = \frac{e^{ix}+e^{-ix}}{2}} =$$

$$= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$$

$$\int\left(\frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}\right)dx = \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x + C$$

c) $\int \sin^5 x\,dx = \int \sin^4 x \cdot \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx =$

$= \begin{bmatrix} \cos x = t \\ dt = -\sin x\,dx \end{bmatrix} = -\int (1-\cos^2 x)^2 \underbrace{(-\sin x)dx}_{=dt} =$

$= -\int (1-t^2)^2\,dt = -\int (1-2t^2+t^4)\,dt =$

$= \int (-1+2t^2-t^4)\,dt = -t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + C =$

$= -\cos x + \frac{2}{3}(\cos x)^3 - \frac{1}{5}(\cos x)^5 + C$

d) $\int \frac{dx}{\cos x} = \begin{bmatrix} \text{se Boken} \\ \text{s 263} \end{bmatrix} = \int \frac{\cos x}{\cos^2 x}\,dx =$

$= \int \frac{\cos x\,dx}{1-\sin^2 x} = \begin{bmatrix} \sin x = t \\ dt = \cos x\,dx \end{bmatrix} =$

$= \int \frac{dt}{1-t^2} = \int \frac{dt}{(1-t)(1+t)} = \begin{bmatrix} \text{partialbråkuppdelning,} \\ \text{använd handpåläggning} \end{bmatrix}$

$= \frac{1}{2}\int \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt =$

$= -\frac{1}{2}\ln|1-t| + \frac{1}{2}\ln|1+t| + C = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C =$

$= \frac{1}{2}\ln \frac{1+\sin x}{1-\sin x} + C$

e) $\int e^{\sin x} \sin 2x\,dx = 2\int e^{\sin x} \sin x \cos x\,dx = \begin{bmatrix} \sin x = t \\ dt = \cos x\,dx \end{bmatrix}$

$= 2\int e^t \cdot t\,dt = 2\left[e^t \cdot t - \int e^t dt\right] = 2e^t \cdot t - e^t + C =$

$= 2e^{\sin x} \cdot \sin x - e^{\sin x} + C$

f) $\int \sin^3 x \cos^4 x \, dx = \int \cos^4 x \cdot \sin^2 x \cdot \sin x \, dx =$

$= \int \cos^4 x \,(1-\cos^2 x) \sin x \, dx = \begin{bmatrix} \cos x = t \\ dt = -\sin x \, dx \end{bmatrix} =$

$= \int t^4 (1-t^2)(-dt) = -\int (t^4 - t^6) \, dt =$

$= -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

23 a) $\int \sin 3x \sin 4x \, dx =$

$= \int \frac{(e^{3ix} - e^{-3ix})}{2i} \cdot \frac{(e^{4ix} - e^{-4ix})}{2i} dx =$

$= -\frac{1}{4} \int (e^{7ix} - e^{ix} - e^{-ix} + e^{-7ix}) \, dx$

$= -\frac{1}{2} \int \left(\frac{e^{7ix} + e^{-7ix}}{2} - \frac{e^{ix} + e^{-ix}}{2} \right) dx =$

$= -\frac{1}{2} \int (\cos 7x - \cos x) \, dx =$

$= -\frac{1}{2} \left[\frac{1}{7} \sin 7x - \sin x \right] + C =$

$= -\frac{1}{14} \sin 7x + \frac{1}{2} \sin x + C$

в) $\int e^x \sin x \, dx = \int e^x \, \frac{e^{ix} - e^{-ix}}{2i} \, dx =$

$= \int \frac{e^{(1+i)x} - e^{(1-i)x}}{2i} \, dx =$

$= \frac{e^{(1+i)x}}{2i(1+i)} - \frac{e^{(1-i)x}}{2i(1-i)} + C =$

$$= \frac{e^{x+ix}}{2i-2} - \frac{e^{x-ix}}{2i+2} + C$$

$$= \frac{e^{x}(\cos x + i\sin x)}{2i-2} - \frac{e^{x}(\cos x - i\sin x)}{2i+2} + C =$$

$$= \frac{e^{x}\left((\cos x + i\sin x)(i+1) - (\cos x - i\sin x)(i-1)\right)}{2(i-1)(i+1)} + C =$$

$$= \frac{e^{x}(\cancel{i\cos x} - \sin x + \cos x + \cancel{i\sin x} - \cancel{i\cos x} - \sin x + \cos x - \cancel{i\sin x})}{2(-1-\cancel{i}+\cancel{i}-1)}$$

$$= \frac{e^{x}(-2\sin x + 2\cos x)}{-4} = \frac{1}{2}e^{x}(-\cos x + \sin x) + C$$

22 $\int \frac{\sin^3 x}{\cos^5 x}\,dx = \int \tan^3 x \cdot \frac{1}{\cos^2 x}\,dx =$

$$= \begin{bmatrix} \tan x = t \\ dt = \frac{dx}{\cos^2 x} \end{bmatrix} = \int t^3 \cdot dt = \frac{t^4}{4} + C =$$

$$= \frac{1}{4}(\tan x)^4 + C = f(x)$$

Eftersom $f(0) = 0 \Rightarrow 0 = \frac{1}{4}(\tan 0)^4 + C \Rightarrow C = 0$

$$\Rightarrow f(x) = \left(\frac{1}{4}\tan x\right)^4 + C$$

Extra

$\boxed{P5}$ **21** i) $\int \sin x \cdot \sin 2x \cdot \sin 3x \, dx =$

$$= \int \left(\left(\frac{e^{ix}-e^{-ix}}{2i}\right) \cdot \left(\frac{e^{3ix}-e^{-3ix}}{2i}\right)\right)\left(\frac{e^{2ix}-e^{-2ix}}{2i}\right) dx =$$

$$= \int \left(\frac{e^{4ix}-e^{2ix}-e^{-2ix}+e^{-4ix}}{-4}\right) \frac{e^{2ix}-e^{-2ix}}{2i} \, dx =$$

$$= \int \frac{e^{6ix}-e^{4ix}-\cancel{1}+e^{-2ix}-e^{2ix}+\cancel{1}+e^{-4ix}-e^{-6ix}}{-8i} \, dx$$

$$= \int \left(\frac{e^{6ix}-e^{-6ix}}{-8i} - \frac{e^{4ix}-e^{-4ix}}{-8i} - \frac{e^{2ix}-e^{-2ix}}{-8i}\right) dx$$

$$= \int \left(-\frac{1}{4}\sin 6x + \frac{1}{4}\sin 4x + \frac{1}{4}\sin 2x\right) dx =$$

$$= + \frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x - \frac{1}{8}\cos 2x + C$$

25 a) Det finns olika metoder, se t ex boken s 264, ex. 5.35. Vi ska dock göra lite annorlunda:

$$\int \frac{25\cos x}{4\cos x + 3\sin x} \, dx \underset{\substack{\text{delar}\\ \text{täl. och}\\ \text{nämn med}\\ \cos x}}{=} \int \frac{25}{4+3\tan x} \, dx =$$

$$= \left[\begin{array}{l} \tan x = t \\ dt = \frac{dx}{\cos^2 x} = (1+\tan^2 x)dx \\ \quad = (1+t^2)\,dx \Rightarrow \\ dx = \frac{dt}{1+t^2} \end{array}\right] = \int \frac{25}{4+3t} \, \frac{dt}{1+t^2} = \otimes$$

Partialbråkuppdelar:

$$\frac{25}{(4+3t)(1+t^2)} = \frac{A}{4+3t} + \frac{Bt+C}{1+t^2}$$

$$25 = A + \underline{At^2} + 4Bt + 4C + \underline{3Bt^2} + 3Ct \Rightarrow$$

$A + 3B = 0$ $\qquad$ $A = -3B$ $\qquad$ $A = 9$

$4B + 3C = 0$ $\qquad$ $C = -\frac{4}{3}B$ $\qquad$ $C = 4$

$4C + A = 25$ $\qquad$ $-\frac{16}{3}B - 3B = 25 \Rightarrow -\frac{25}{3}B = 25 \Rightarrow B = -3$

$$\otimes = \int \frac{9}{4+3t}\,dt + \int \frac{-3t+4}{1+t^2}\,dt =$$

$$= 3\int \frac{3}{4+3t}\,dt - \frac{3}{2}\int \frac{2t}{1+t^2}\,dt + 4\int \frac{dt}{1+t^2} =$$

$$= 3\ln|4+3t| - \frac{3}{2}\ln|1+t^2| + 4\arctan t + C =$$

$$\arctan(\tan x) = x + \pi n$$

$$= 3\ln|4+3\tan x| - 3\ln\sqrt{1+\tan^2 x} + 4x + C$$

$$= 3\ln\left|\frac{4\cos x + 3\sin x}{\cos x}\right| - 3\ln\left|\frac{1}{\cos x}\right| + 4x + C =$$

$$= 3\ln\left|\frac{\frac{4\cos x + 3\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}}\right| + 4x + C =$$

$$= 3\ln|4\cos x + 3\sin x| + 4x + C.$$

b) Vi använder

$$\sin^3 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{-8i} =$$

$$= \frac{e^{3ix} - e^{-3ix}}{-8i} - 3\,\frac{e^{ix} - e^{-ix}}{-8i} = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x \Rightarrow$$

$$\int x\sin^3 x\,dx = -\frac{1}{4}\int \overset{\downarrow}{x}\,\overset{\uparrow}{\sin 3x}\,dx + \frac{3}{4}\int \overset{\downarrow}{x}\,\overset{\uparrow}{\sin x}\,dx =$$

$$= -\frac{1}{4}\left[-x\frac{\cos 3x}{3} + \int \frac{\cos 3x}{3}\,dx\right] +$$

$$+ \frac{3}{4}\left[-x\cos x + \int \cos x\,dx\right] =$$

$$= \frac{x\cos 3x}{12} - \frac{\sin 3x}{36} - \frac{3x\cos x}{4} + \frac{3}{4}\sin x + C$$

$$= \frac{x}{12}(\cos 3x - 9\cos x) + \frac{1}{36}(27\sin x - \sin 3x) + C$$