Lektion 13

$\boxed{P5}$

1 a) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \dfrac{x^{3/2}}{\frac{3}{2}} + C = \dfrac{2x\sqrt{x}}{3} + C$

b) $\int (x^3 + x - 2)\,dx = \int x^3\,dx + \int x\,dx - 2\int dx =$

$= \dfrac{x^4}{4} + \dfrac{x^2}{2} - 2x + C$

c) $\int \left(\dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{1}{x^3} \right) dx = \int (x^{-1} - x^{-2} + x^{-3})\,dx =$

$= \ln|x| - \dfrac{x^{-1}}{-1} + \dfrac{x^{-2}}{-2} + C =$

$= \ln|x| + \dfrac{1}{x} - \dfrac{1}{2x^2} + C.$

d) $\int \dfrac{dx}{(x+2)^2} = ?$

Eftersom $\int \dfrac{1}{x^2}\,dx = \dfrac{x^{-1}}{-1} = -\dfrac{1}{x}$,

så borde $\int \dfrac{dx}{(x+2)^2}$ vara ungefär $-\dfrac{1}{(x+2)}$.

Men derivatan $\left(-\dfrac{1}{(x+2)}\right)' = \dfrac{1}{(x+2)^2}$, så

$\int \dfrac{dx}{(x+2)^2} = -\dfrac{1}{x+2} + C$

för att beskriva alla primitiva funktioner

e) $\int (x-2)\sqrt{x}\,dx = \int \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right) dx =$

$= \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2x^2\sqrt{x}}{5} - \dfrac{4x\sqrt{x}}{3} + C.$

f) $\int e^{-x}\,dx = ?$

$\int e^{x}\,dx = e^{x}$ så $\int e^{-x}\,dx$ borde vara ungefär e^{-x}. Men

$(e^{-x})' = e^{-x}\cdot(-1) = -e^{-x}$ eller

$(-e^{-x})' = e^{-x} \Rightarrow \int e^{-x}\,dx = -e^{-x} + C$.

g) $\int \frac{dx}{1+4x^2} - ?$ $\int \frac{1}{1+x^2}\,dx = \arctan x$, så

$\int \frac{dx}{1+(2x)^2}$ borde vara ungefär $\arctan 2x$.

Men $(\arctan 2x)' = \frac{1}{1+(2x)^2}\cdot 2 = \frac{2}{1+4x^2}$,

eller $\left(\frac{1}{2}\arctan 2x\right)' = \frac{1}{1+4x^2}$, så

$\int \frac{dx}{1+4x^2} = \frac{1}{2}\arctan 2x + C$

h) $\int \sin 2x\,dx - ?$ $\int \sin x = -\cos x$, så

$\int \sin 2x\,dx$ borde vara ungefär $-\cos 2x$.

Men $(-\cos 2x)' = -(-\sin 2x)\cdot 2 = 2\sin 2x$

eller $\left(-\frac{1}{2}\cos 2x\right)' = \sin 2x$, så

$\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$.

2 I denna uppgift kan man använde (istället att gissa) variabelsubstitution

a) $\int x(1+x^2)^5 dx = \left[\begin{array}{l} 1+x^2 = y \Rightarrow d(1+x^2) = d(y) \Leftrightarrow \\ (x^2)' dx = (y)' dy \Leftrightarrow 2x\,dx = dy \end{array}\right] =$

$= \int (\underbrace{1+x^2}_{=y})^5 \cdot \frac{2x\,dx}{2} = \frac{1}{2}\int y^5 dy =$

$= \frac{1}{2}\frac{y^6}{6} + C = \frac{(1+x^2)^6}{12} + C.$

Metod 2 $\quad \left((1+x^2)^6\right)' = 6(1+x^2)^5 \cdot 2x =$

$= 12x(1+x^2)^5 \Rightarrow$

$x(1+x^2)^5 = \left(\frac{1}{12}(1+x^2)^6\right)'$

$\Rightarrow \int x(1+x^2)^5 dx = \frac{1}{12}(1+x^2)^6 + C$

b) $\int x e^{x^2} dx = \left[\begin{array}{l} x^2 = y \Rightarrow d(x^2) = dy \Rightarrow \\ \Rightarrow 2x\,dx = dy \end{array}\right] =$

$= \int e^y \cdot \frac{1}{2} dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2} + C$

c) $\int \frac{\ln x}{x} dx = \left[\ln x = y \Rightarrow dy = d(\ln x) = \frac{1}{x}dx\right] =$

$= \int y\,dy = \frac{y^2}{2} + C = \frac{(\ln x)^2}{2} + C$

d) $\int \frac{e^x}{2+e^x} dx = \left[\begin{array}{l} 2+e^x = y \\ dy = d(2+e^x) = e^x dx \end{array}\right] =$

$= \int \frac{dy}{y} = \ln|y| + C = \ln(2+e^x) + C$

3 a) $f'(x) = e^{2x} + x^2 - x = \left[(e^{2x})' = 2e^{2x} \right]$

$$= \left(\frac{1}{2}e^{2x}\right)' + \left(\frac{x^3}{3}\right)' - \left(\frac{x^2}{2}\right)' =$$

$$= \left(\frac{1}{2}e^{2x} + \frac{x^3}{3} - \frac{x^2}{2}\right)' \Rightarrow$$

$$f(x) = \frac{1}{2}e^{2x} + \frac{x^3}{3} - \frac{x^2}{2} + C.$$

$f(0) = 0$ ger $\quad \frac{1}{2} \cdot 1 + C = 0 \Rightarrow C = -\frac{1}{2}.$

Så $\quad f(x) = \frac{1}{2}e^{2x} + \frac{x^3}{3} - \frac{x^2}{2} - \frac{1}{2}.$

b) $f(x) = \int \frac{x}{(2+3x^2)^3}\,dx = \left[\begin{array}{l} 2+3x^2 = y \\ dy = (2+3x^2)'dx = \\ \quad = 6x\,dx \end{array} \right]$

$$= \int \frac{\overbrace{6x\,dx}^{=dy}}{6(2+3x^2)^3} = \frac{1}{6}\int \frac{dy}{y^3} =$$

$$= \frac{1}{6} \cdot \frac{y^{-2}}{-2} + C = -\frac{1}{12y^2} + C =$$

$$= -\frac{1}{12(2+3x^2)^2} + C.$$ Observera nu att

$$1 = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\underbrace{-\frac{1}{12(2+3x^2)^2}}_{\to 0} + C\right)$$

$$= C \Rightarrow C = 1$$

Så $\quad f(x) = -\frac{1}{12(2+3x^2)^2} + 1$

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$$\int (4x^2 - 4x + 6) e^{-2x} dx =$$

$$= \left[\begin{array}{l} y = -2x \,,\; x = -\frac{1}{2} y \\ dy = (-2x)' dx = -2\,dx \end{array} \right] =$$

$$= \int \left((-2x)^2 + 2 \cdot (-2x) + 6 \right) \cdot e^{-2x} \cdot \underbrace{\frac{-2}{-2} dx}_{= dy}$$

$$= -\frac{1}{2} \int (y^2 + 2y + 6) e^y dy =$$

$$= -\frac{1}{2} \int (y^2 + 2y) e^y dy - 3 \int e^y dy$$

$$= -\frac{1}{2} \int (y^2 + 2y) e^y dy - 3e^y = \otimes$$

För att beräkna $\int (y^2 + 2y) e^y dy$, observera att $(y^2 \cdot e^y)' = 2y\, e^y + y^2 e^y = (y^2 + 2y) e^y$, så

$$\otimes = -\frac{1}{2} y^2 \cdot e^y - 3e^y + C =$$

$$= -\frac{1}{2} \cdot 4x^2 \cdot e^{-2x} - 3e^{-2x} + C =$$

$$= (-2x^2 - 3) e^{-2x} + C.$$

7 Om F och G är primitiva funktioner till f då är

$$\begin{array}{l} F'(x) = f(x) \\ G'(x) = f(x) \end{array} \Rightarrow F'(x) - G'(x) = 0.$$

Detta betyder att $(F(x) - G(x))' = 0 \Rightarrow$
$F(x) - G(x) = C \Rightarrow F(x) = G(x) + C$.

$\boxed{B5}$ **25** Eftersom

$$\left(\ln\left|x+\sqrt{x^2+a}\right|+C\right)' =$$

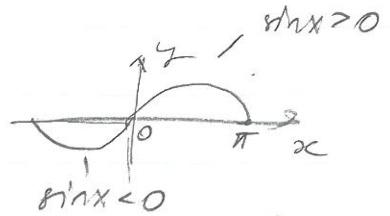
$$= \frac{1}{x+\sqrt{x^2+a}}\cdot\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a}}\right) =$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+a}}}\cdot\frac{\cancel{\sqrt{x^2+a}+x}}{\sqrt{x^2+a}} = \frac{1}{\sqrt{x^2+a}},$$

är $\displaystyle\int\frac{dx}{\sqrt{x^2+a}} = \ln\left|x+\sqrt{x^2+a}\right|+C.$

Extra

$\boxed{P5}$ **32**



$$|\sin x| = \begin{cases}\sin x & 0\le x\le\pi\\ -\sin x & -\pi\le x<0\end{cases}$$

$\int\sin x\,dx = -\cos x + C_1$,

$\int(-\sin x)\,dx = \cos x + C_2$

så $\displaystyle\int|\sin x|\,dx = \begin{cases}-\cos x + C_1 & 0\le x\le\pi\\ \cos x + C_2 & -\pi\le x<0\end{cases}$

Men $f'(x) = |\sin x|$, så $f(x) = \int|\sin x|\,dx$ måste vara deriverbar, speciellt kontinuerlig.

Då $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^-} f(x)$ måste vara sant,

dvs $-1 + C_1 = 1 + C_2 \Rightarrow C_2 = -2 + C_1$

Vi ser att $f(x) = \begin{cases}-\cos x + C_1 & 0\le x\le\pi\\ \cos x + C_1 - 2 & -\pi\le x<0\end{cases}$,

och det är lätt att se att $f'_+(0) = \sin x|_{x=0} =$

$= f'_-(0) = -\sin x|_{x=0}$. $\longrightarrow$

Eftersom $f(-\pi)=0 \Rightarrow 0=\underbrace{\cos(-\pi)}_{=-1}+C_1-2 \Rightarrow \underline{\underline{C_1=3}}$

Så $f(x)=\begin{cases} 3-\cos x & 0\le x\le\pi \\ 1+\cos x & -\pi\le x\le 0 \end{cases}$.

Som följd, $f(\pi)=3-\cos\pi=\underline{\underline{4}}$.